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Differential equations are fundamental tools in mathematics, physics, engineering, and many scientific disciplines. They describe how quantities change and interact over time or space, providing insights into the behavior of systems ranging from simple mechanical motions to complex biological processes. One common type of differential equation encountered in various applications is the first-order differential equation, particularly those that are exact. Recognizing whether a differential equation is exact and solving it efficiently is an essential skill for students and professionals alike.

In this article, we explore the concept of exact differential equations, how to determine if a given differential equation qualifies as exact, and the step-by-step process to solve those that are. Additionally, we address what to do if a differential equation is not exact and how to manipulate it into an exact form or alternative solution strategies. By the end of this guide, you will have a comprehensive understanding of the criteria and methods involved in working with exact differential equations, along with practical examples to solidify your grasp.

Understanding Differential Equations: An Overview

Before diving into the specifics of exact equations, it's important to understand the general framework of differential equations.

What is a Differential Equation?

A differential equation involves an unknown function and its derivatives. It expresses the relationship between the function's values and rates of change. For example, a simple first-order differential equation can be written as:

$$
\frac{dy}{dx} = f(x, y)
$$

where $y = y(x)$ is the unknown function.

Types of Differential Equations

Differential equations are classified based on order, linearity, and other properties:

- Order: The highest derivative present (first-order, second-order, etc.)
- Linearity: Whether the equation is linear or nonlinear
- Exactness: Whether the differential form can be integrated directly

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What Is an Exact Differential Equation?

An exact differential equation is a special form of a first-order differential equation that can be directly integrated to find the solution without additional manipulation.

Form of an Exact Differential Equation

A first-order differential equation is said to be exact if it can be written in the form:

$$
M(x, y) \, dx + N(x, y) \, dy = 0
$$

where M and N are functions of x and y.

The key property of an exact differential equation is that there exists a potential function $\Psi(x, y)$ such that:

$$
\frac{\partial \Psi}{\partial x} = M(x, y), \quad \frac{\partial \Psi}{\partial y} = N(x, y)
$$

and the differential equation corresponds to the total differential:

$$
d\Psi = \frac{\partial \Psi}{\partial x} dx + \frac{\partial \Psi}{\partial y} dy = 0
$$

How to Determine if a Differential Equation Is Exact

To check whether a differential equation is exact, follow these steps:

Step 1: Write the Equation in Standard Form

Express the differential equation in the form:

$$
M(x, y) \, dx + N(x, y) \, dy = 0
$$

Ensure that the equation is explicitly arranged with dx and dy.

Step 2: Calculate Partial Derivatives

Compute:

$$
\frac{\partial M}{\partial y} \quad \text{and} \quad \frac{\partial N}{\partial x}
$$

Step 3: Compare the Partial Derivatives

- If

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

then the differential equation is exact.

- If not, the equation is not exact, and you may need to find an integrating factor or apply other methods.

Summary of the Test for Exactness

Step	Operation	Condition for Exactness
1	Write in form $M dx + N dy = 0$	Equation is in the correct form
2	Compute $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$	Check if they are equal
3	Verify if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$	If yes, equation is exact

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Solving an Exact Differential Equation

Once confirmed that the differential equation is exact, follow these steps to find the solution:

Step 1: Find the Potential Function $\Psi(x, y)$

- Integrate $M(x, y)$ with respect to x:

$$
\Psi(x, y) = \int M(x, y) \, dx + h(y)
$$

where $h(y)$ is an arbitrary function of y (since the partial derivative with respect to x eliminates $h(y)$).

Step 2: Determine $h(y)$

- Differentiate $\Psi(x, y)$ with respect to y:

$$
\frac{\partial \Psi}{\partial y} = \frac{\partial}{\partial y} \left(\int M \, dx + h(y) \right) = N(x, y)
$$

- Solve for $h'(y)$:

$$
h'(y) = N(x, y) - \frac{\partial}{\partial y} \left(\int M \, dx \right)
$$

- Integrate $h'(y)$ with respect to y to find $h(y)$.

Step 3: Write the General Solution

The general solution is given implicitly by:

$$
\Psi(x, y) = C
$$

where C is an arbitrary constant.

Practical Example: Solving an Exact Differential Equation

Suppose we are given the differential equation:

$$
(2xy + y^2) \, dx + (x^2 + 2xy) \, dy = 0
$$

Let's determine if it is exact and solve it if it is.

Step 1: Write in Standard Form

Identify:

$$
M(x, y) = 2xy + y^2, \quad N(x, y) = x^2 + 2xy
$$

Step 2: Compute Partial Derivatives

$$
\frac{\partial M}{\partial y} = 2x + 2y
$$

$$
\frac{\partial N}{\partial x} = 2x + 2y
$$

Since:

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2x + 2y
$$

the equation is exact.

Step 3: Find the Potential Function $\Psi(x, y)$

- Integrate M with respect to x:

$$
\Psi(x, y) = \int (2xy + y^2) \, dx = y \int 2x \, dx + y^2 \int dx = y (x^2) + y^2 x + h(y)
$$

which simplifies to:

$$
\Psi(x, y) = x^2 y + xy^2 + h(y)
$$

- Differentiate Ψ with respect to y:

$$
\frac{\partial \Psi}{\partial y} = x^2 + 2xy + h'(y)
$$

Set equal to $N(x, y)$:

$$
x^2 + 2xy + h'(y) = x^2 + 2xy
$$

- Solve for $h'(y)$:

$$
h'(y) = 0
$$

- Integrate to find $h(y)$:

$$
h(y) = \text{constant}
$$

Final Solution:

$$
\Psi(x, y) = x^2 y + xy^2 = C
$$

which implicitly defines the solution to the differential equation.

What If the Differential Equation Is Not Exact?

Sometimes, a differential equation does not satisfy the exactness condition. In such cases, you can try one of the following approaches:

Method 1: Find an Integrating Factor

- An integrating factor is a function $\mu(x, y)$ that, when multiplied with the original differential equation, makes it exact.
- Common choices include functions of x alone or y alone, such as $\mu(x)$ or $\mu(y)$.
- To find such an integrating factor, analyze the derivatives and use known criteria or formulas.