Find The Coordinates Of The Point (x, Y, Z) On The Plane Z = 4 X +2 Y + 3 Which Is Closest To The Origin.x

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Introduction

In geometry and vector calculus, the problem of finding the point on a given surface or plane that is closest to a specific point—commonly the origin—is fundamental. Here, we are asked to determine the coordinates of the point (x, y, z) on the plane $z = 4x + 2y + 3$ that is closest to the origin $(0, 0, 0)$. This problem involves understanding the geometric relationship between the plane and the point, and applying optimization techniques to find the minimal distance.

The solution involves translating the geometric problem into an optimization problem, where the goal is to minimize the distance from the origin to any point on the plane. This typically involves using the distance formula and methods from calculus or linear algebra, such as the method of Lagrange multipliers or projections onto the plane.

Understanding the Geometric Setup

Equation of the Plane

The plane is given by the equation:

$$
z = 4x + 2y + 3
$$

which can also be written in the standard form:

$$
4x + 2y - z + 3 = 0
$$

This is a linear plane in three-dimensional space.

Point of Reference: The Origin

The point from which we measure the shortest distance is the origin:

$$
O = (0, 0, 0)
$$

Our goal is to find a point $P = (x, y, z)$ on the plane such that the Euclidean distance:

$$
d = \sqrt{(x - 0)^2 + (y - 0)^2 + (z - 0)^2}
$$

is minimized.

Mathematical Formulation of the Problem

Objective Function

The distance from the origin to a general point (x, y, z) on the plane is:

$$
d(x, y) = \sqrt{x^2 + y^2 + z^2}
$$

Since the square root is a monotonically increasing function, minimizing d is equivalent to minimizing its square:

$$
D(x, y) = x^2 + y^2 + z^2
$$

Using the plane's equation to express z:

$$
z = 4x + 2y + 3
$$

we substitute into $D(x, y)$:

$$
D(x, y) = x^2 + y^2 + (4x + 2y + 3)^2
$$

Our task reduces to finding the minimum of $D(x, y)$.

Optimization Approach

Method 1: Direct Substitution and Calculus

We can treat $D(x, y)$ as a function of two variables and find its critical points by taking partial derivatives.

Step 1: Write the function explicitly

$$
D(x, y) = x^2 + y^2 + (4x + 2y + 3)^2
$$

Expanding the squared term:

$$
(4x + 2y + 3)^2 = 16x^2 + 16xy + 9 + 12x + 12y + 4y^2
$$

But to be precise, expand carefully:

$$
(4x + 2y + 3)^2 = (4x)^2 + 2 \times 4x \times 2y + (2y)^2 + 2 \times 4x \times 3 + 2 \times 2y \times 3 + 3^2
$$

Simplify term-by-term:

- $(4x)^2 = 16x^2$
- $2 \times 4x \times 2y = 16xy$
- $(2y)^2 = 4y^2$
- $2 \times 4x \times 3 = 24x$
- $2 \times 2y \times 3 = 12y$
- $3^2= 9$

Thus,

$$
(4x + 2y + 3)^2 = 16x^2 + 16xy + 4y^2 + 24x + 12y + 9
$$

Now, substitute back into $D(x, y)$:

$$
D(x, y) = x^2 + y^2 + 16x^2 + 16xy + 4y^2 + 24x + 12y + 9
$$

Combine like terms:

$$
D(x, y) = (x^2 + 16x^2) + y^2 + 4y^2 + 16xy + 24x + 12y + 9
$$

$$
D(x, y) = 17x^2 + 5y^2 + 16xy + 24x + 12y + 9
$$

\]

Step 2: Find critical points by setting partial derivatives to zero

Compute the partial derivatives:

$$
\frac{\partial D}{\partial x} = 34x + 16y + 24
$$
$$
\frac{\partial D}{\partial y} = 10y + 16x + 12
$$

Set both derivatives to zero:

$$
34x + 16y + 24 = 0 \quad \text{(1)}
$$
$$
10y + 16x + 12 = 0 \quad \text{(2)}
$$

Step 3: Solve the system of equations

From equation (2):

$$
10y + 16x = -12
$$
$$
\Rightarrow y = \frac{-12 - 16x}{10} = -\frac{6}{5} - \frac{8}{5}x
$$

Substitute into equation (1):

$$
34x + 16 \left(-\frac{6}{5} - \frac{8}{5}x \right) + 24 = 0
$$

Simplify:

$$
34x - \frac{96}{5} - \frac{128}{5}x + 24 = 0
$$

Multiply through by 5 to clear denominators:

$$
5 \times 34x - 96 - 128x + 5 \times 24 = 0
$$

$$
170x - 96 - 128x + 120 = 0
$$

Combine like terms:

$$
(170x - 128x) + (-96 + 120) = 0
$$
$$
42x + 24 = 0
$$
$$
42x = -24
$$
$$
x = -\frac{24}{42} = -\frac{4}{7}
$$

Now, substitute $x = -\frac{4}{7}$ back into the expression for y:

$$
y = -\frac{6}{5} - \frac{8}{5} \times \left(-\frac{4}{7} \right) = -\frac{6}{5} + \frac{8 \times 4}{5 \times 7} = -\frac{6}{5} + \frac{32}{35}
$$

Express $-\frac{6}{5}$ as $\frac{-42}{35}$:

$$
y = -\frac{42}{35} + \frac{32}{35} = -\frac{10}{35} = -\frac{2}{7}
$$

Finally, compute z:

$$
z = 4x + 2y + 3
$$

Substitute $x = -\frac{4}{7}$, $y = -\frac{2}{7}$:

$$
z = 4 \times \left(-\frac{4}{7} \right) + 2 \times \left(-\frac{2}{7} \right) + 3 = -\frac{16}{7} - \frac{4}{7} + 3
$$

Express 3 as $\frac{21}{7}$:

$$
z = -\frac{16}{7} - \frac{4}{7} + \frac{21}{7} = \frac{-16 - 4 + 21}{7} = \frac{1}{7}
$$

Result: Coordinates of the Closest Point

The point (x, y, z) on the plane closest to the origin is:

$$
\boxed{
\left(-\frac{4}{7}, -\frac{2}{7}, \frac{1}{7} \right)
}
$$

Verification and Geometric Interpretation

Distance Calculation