Find The General Solution Of The Given Differential Equation (you Can Use Either Undetermined Coefficients

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Understanding how to find the general solution of a differential equation is fundamental in calculus and applied mathematics. Whether you're analyzing physical systems, modeling economic behavior, or solving engineering problems, mastering methods to solve differential equations provides valuable insights into the dynamics of various phenomena. Among these methods, the Undetermined Coefficients technique stands out for its effectiveness in solving linear differential equations with constant coefficients, especially those with nonhomogeneous terms such as polynomials, exponentials, sines, and cosines. This comprehensive guide will walk you through the process of finding the general solution of such differential equations, illustrating the method with clear explanations, examples, and best practices.

Understanding Differential Equations and Their Types

Before diving into the solution methods, it's essential to understand what differential equations are and how they are categorized.

What Is a Differential Equation?

A differential equation is an equation that involves an unknown function and its derivatives. It relates the function's rates of change to the function itself or other variables.

Example:

$$ \frac{dy}{dx} + 3y = 6x $$

Here, the derivative of y with respect to x appears in the equation, indicating that this is a differential equation.

Types of Differential Equations

Differential equations can be classified based on various criteria:

- **Order:** The highest derivative present determines the order. For example, if the highest derivative is $\frac{d^2y}{dx^2}$, the equation is second-order.

- **Linearity:** If the unknown function and its derivatives appear linearly (no powers, products, or nonlinear functions), the equation is linear.

- **Homogeneity:** If the nonhomogeneous term (forcing function) is zero, the equation is homogeneous; otherwise, nonhomogeneous.

In this guide, we focus on linear, constant-coefficient differential equations, which are most suitable for the Undetermined Coefficients method.

Standard Form of Linear Differential Equations with Constant Coefficients

A typical second-order linear differential equation with constant coefficients takes the form:

$$
a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = f(x)
$$

where:
- a, b, c are constants.
- $f(x)$ is a known function representing the nonhomogeneous part.

The goal is to find the general solution, which is the sum of the complementary (homogeneous) solution and the particular solution:

$$
y_{gen} = y_h + y_p
$$

Step-by-Step Approach to Find the General Solution

The process involves several key steps:

1. **Solve the homogeneous equation** to find the complementary solution y_h.

2. **Identify the form of the nonhomogeneous term $f(x)$** to decide on the particular solution y_p.

3. **Determine the particular solution y_p** using the Undetermined Coefficients method.

4. **Combine solutions** to write the general solution: $y_{gen} = y_h + y_p$.

Step 1: Solving the Homogeneous Equation

The homogeneous differential equation associated with a given linear differential equation is obtained by setting the nonhomogeneous term $f(x)$ to zero:

$$
a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0
$$

Procedure:

1. Write the characteristic equation: $a r^2 + b r + c = 0$.

2. Find the roots r_1 and r_2 of the quadratic equation.

3. Form the homogeneous solution based on the roots:

    - If roots are real and distinct: $y_h = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

    - If roots are real and repeated: $y_h = (C_1 + C_2 x) e^{r x}$

    - If roots are complex conjugates: $r = \alpha \pm \beta i$, then $y_h = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$

Example:

Solve $y'' - 3 y' + 2 y = 0$.

- Characteristic equation: $r^2 - 3 r + 2 = 0$
- Roots: $r = 1, 2$
- Homogeneous solution: $y_h = C_1 e^{x} + C_2 e^{2x}$

Step 2: Analyzing the Nonhomogeneous Term $f(x)$

The form of $f(x)$ guides the choice of the particular solution y_p. Common forms include:

- Polynomials: $f(x) = P_n(x)$

- Exponentials: $f(x) = A e^{k x}$

- Sines and cosines: $f(x) = B \sin \omega x + C \cos \omega x$

- Products of the above (e.g., $x e^{k x}$, $x \sin \omega x$)

Important: If $f(x)$ resembles the complementary solution, adjustments must be made to avoid duplication, often by multiplying the particular solution guess by x.

Step 3: Using the Method of Undetermined Coefficients

This method involves guessing the form of y_p based on $f(x)$ and then determining the unknown coefficients.

General Strategy:

1. Guess the particular solution y_p based on the form of $f(x)$.
2. Substitute y_p into the differential equation.
3. Solve for the unknown coefficients by equating coefficients of like terms.

Common Forms of y_p:

$f(x)$ Type	Guess for y_p	Notes
Polynomial $P_n(x)$	$y_p = A_0 + A_1 x + \dots + A_n x^n$	Adjust if y_h contains polynomials of the same degree
Exponential $A e^{k x}$	$y_p = B e^{k x}$	Multiply by x if $e^{k x}$ is part of y_h
Sine or cosine $B \sin \omega x + C \cos \omega x$	$y_p = D \sin \omega x + E \cos \omega x$	Multiply by x if necessary
Product forms, e.g., $x e^{k x}$	$y_p = (A x + B) e^{k x}$	Adjust based on duplication with \(

y_h \) |

Step 4: Adjusting for Repeated Roots or Terms

If the guessed particular solution overlaps with the homogeneous solution (i.e., the form of y_p is part of y_h), multiply the guess by x (or higher powers of x if necessary) to obtain a valid particular solution.

Example:
Suppose $f(x) = e^{2x}$ and the homogeneous solution already contains e^{2x}. Then, guess:

$$
y_p = x A e^{2x}
$$

Instead of $y_p = A e^{2x}$.

Step 5: Solving for Coefficients and Writing the General Solution

1. Substitute the guessed y_p into the original differential equation.
2. Differentiate as required.
3. Collect like terms and equate coefficients to solve for the unknowns.
4. Write the particular solution y_p.

Finally, combine y_h and y_p:

$$
\boxed{
y_{gen} = y_h + y_p
}
$$

This represents the most general form of the solution, encompassing all possible solutions to the differential equation.

Worked Example: Applying the Method to a Specific Equation

Let's illustrate the process with a concrete example.

Problem:

Solve the differential equation:

$$
y'' +
$$

Frequently Asked Questions

What is the general approach to solving a differential equation using the method of undetermined coefficients?

The method involves finding a particular solution by guessing a form based on the nonhomogeneous term, then determining the coefficients by substituting into the differential equation. The general solution is the sum of the complementary (homogeneous) solution and the particular solution.

When can I use the method of undetermined coefficients to find the general solution?

You can use this method when the differential equation is linear with constant coefficients and the nonhomogeneous term is of a specific form such as polynomials, exponentials, sines, or cosines, or their combinations.

How do I choose the form of the particular solution in the method of undetermined coefficients?

You select the form of the particular solution based on the nonhomogeneous term. For example, if the RHS is a polynomial, try a polynomial; if it's exponential, try an exponential function, and similarly for sines and cosines.

What should I do if the guessed particular solution overlaps with the homogeneous solution?

If the guessed particular solution is similar to the homogeneous solution, multiply your guess by x (or a higher power if necessary) to ensure linear independence and find the particular solution accordingly.

Can the undetermined coefficients method be used for non-constant coefficient differential equations?

No, the method of undetermined coefficients is primarily applicable to linear differential equations with constant coefficients. For variable coefficients, methods like variation of parameters are more appropriate.

What are common forms of the nonhomogeneous term for which the undetermined coefficients method is effective?

Common forms include polynomials, exponential functions, sine and cosine functions, or sums of these. These allow for straightforward guessing of the particular solution form.

How do I verify that my general solution is correct after applying the undetermined coefficients method?

Substitute the general solution back into the original differential equation to check if it satisfies the equation. Simplify to confirm that both sides are equal, ensuring the correctness of your solution.

Additional Resources

Find The General Solution Of The Given Differential Equation (You Can Use Either Undetermined Coefficients)

Introduction

Differential equations are fundamental in mathematical modeling across science, engineering, economics, and numerous other fields. They describe how a quantity changes concerning another, often time or space, and their solutions provide deep insights into the behavior of complex systems. Among various methods to solve differential equations, the method of Undetermined Coefficients is a particularly elegant and practical technique, especially for linear differential equations with constant coefficients and certain types of nonhomogeneous terms.

In this comprehensive review, we will explore the process of finding the general solution of a differential equation using the Undetermined Coefficients method. We will delve into the theoretical foundations, step-by-step procedures, application examples, and common pitfalls, ensuring a thorough understanding of the topic.

Understanding the Differential Equation

Before delving into solution methods, it's crucial to understand the structure of the differential equation:

- Type of Differential Equation:
- Linear or nonlinear? The method of undetermined coefficients applies primarily to linear differential equations.
- Order of the differential equation: First order, second order, or higher.
- Homogeneous or nonhomogeneous? The method is designed for nonhomogeneous equations, where the right-hand side (forcing function) is non-zero.

- Form of the Differential Equation:
- Typically, equations are expressed as:

$$
a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 \frac{dy}{dx} + a_0 y = g(x)
$$

where a_i are constants, and $g(x)$ is a known function.

The General Solution: Homogeneous + Particular

The general solution of a linear nonhomogeneous differential equation can be expressed as:

$$
y(x) = y_h(x) + y_p(x)
$$

where:

- $y_h(x)$ is the complementary (homogeneous) solution, obtained by solving the associated homogeneous equation:

$$
a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 \frac{dy}{dx} + a_0 y = 0
$$

- $y_p(x)$ is the particular solution, found via the method of Undetermined Coefficients.

Homogeneous Solution: The Foundation

Solving the Homogeneous Equation

The homogeneous solution involves:

1. Formulating the characteristic equation:

$$
a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0 = 0
$$

2. Finding roots:

- Real and distinct roots: straightforward exponential solutions.
- Repeated roots: multiply by powers of x.
- Complex roots: involve sinusoidal functions.

3. Constructing the homogeneous solution y_h:

Based on roots, the general form of y_h is constructed as a linear combination of exponential, sine, and cosine functions.

Particular Solution via Undetermined Coefficients

The core focus of this review is on finding y_p using the Undetermined Coefficients method.

When to Use the Method

This method is applicable when:

- The differential equation is linear with constant coefficients.
- The nonhomogeneous term $g(x)$ is of a particular class, such as:
- Polynomials
- Exponentials
- Sines and cosines
- Combinations of these functions

- The form of $g(x)$ is such that the guess for y_p can be systematically constructed.

Step-by-Step Procedure for Undetermined Coefficients

1. Identify the form of \(g(x) \)

Determine the type of nonhomogeneous term:

- Polynomial: \(P(x) \)
- Exponential: \(e^{ax} \)
- Sinusoidal: \(\sin bx \) or \(\cos bx \)
- Combinations: e.g., \(P(x) e^{ax} \), \(P(x) \sin bx \), etc.

2. Guess the particular solution \(y_p \)

Construct an ansatz (initial guess) based on the form of \(g(x) \):

- For polynomials of degree \(n \):

\[
y_p = A_0 + A_1 x + A_2 x^2 + \dots + A_n x^n
\]

- For exponentials \(e^{ax} \):

\[
y_p = A e^{ax}
\]

- For sines and cosines \(\sin bx \) or \(\cos bx \):

\[
y_p = A \sin bx + B \cos bx

\]

- For products like $P(x) e^{ax}$, or $P(x) \sin bx$:

- Use polynomial times exponential or sinusoid as the form, and include polynomial terms as needed.

3. Adjust the guess if it overlaps with the homogeneous solution

If the guessed form is similar to the homogeneous solution (i.e., the roots of the characteristic equation are related to the form of $g(x)$), multiply the guess by x (or x^k) until the particular solution is linearly independent from the homogeneous solution.

4. Substitute into the differential equation

Compute derivatives of y_p, substitute into the original differential equation, and solve for the undetermined coefficients.

5. Solve for the coefficients

Set up algebraic equations by equating coefficients of like terms, then solve for the unknowns $A_i, B,$ etc.

Examples of Applying Undetermined Coefficients

Example 1: Polynomial Forcing Function

Solve:

\[

$$
y'' - 3 y' + 2 y = 4x^2
$$

Solution:

- Homogeneous solution:

$$
y_h: \quad r^2 - 3r + 2=0 \Rightarrow (r-1)(r-2)=0
$$

$$
y_h = C_1 e^{x} + C_2 e^{2x}
$$

- Particular solution guess:

Since RHS is quadratic polynomial $4x^2$, try:

$$
y_p = A x^2 + B x + C
$$

- Compute derivatives:

$$
y_p' = 2A x + B
$$

$$
y_p''= 2A
$$

\]

- Substitute into the differential equation:

\[
2A - 3(2A x + B) + 2(A x^2 + B x + C) = 4x^2
\]

- Simplify:

\[
2A - 6A x - 3B + 2A x^2 + 2B x + 2C = 4x^2
\]

- Group like terms:

\[
(2A x^2) + (-6A x + 2B x) + (2A - 3B + 2C) = 4x^2
\]

- Equate coefficients:

- For \(x^2 \):

\[
2A = 4 \Rightarrow A=2
\]

- For \(x \):

\[

$$
-6A + 2B = 0 \Rightarrow -12 + 2B = 0 \Rightarrow 2B=12 \Rightarrow B=6
$$

- For constants:

$$
2A - 3B + 2C = 0 \Rightarrow 4 - 18 + 2C= 0 \Rightarrow 2C=14 \Rightarrow C=7
$$

- Particular solution:

$$
y_p = 2x^2 + 6x + 7
$$

- General solution:

$$
y = y_h + y_p = C_1 e^{x} + C_2 e^{2x} + 2x^2 + 6x + 7
$$

Example 2: Exponential Forcing Function

Solve:

$$
y'' + y = e^{2x}
$$

Solution:

- Homogeneous:

\[

r^2 + 1 = 0 \Rightarrow r= \pm i

\]

\[

y_h = C_1 \cos x + C_2 \sin x

\]

- Since RHS is \(e^{2x} \), guess:

\[

y_p = A e^{2x}

\]

- Derivatives:

\[

y

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