First Exam Question 3 : Determine And Sketch The Response Y(t) Of The LTI System With The Impulse Response

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Understanding the behavior of Linear Time-Invariant (LTI) systems is fundamental in signals and systems analysis, control systems, and electrical engineering. The third question in the first exam typically involves determining the output response $Y(t)$ of an LTI system when the system's impulse response $h(t)$ is known, along with a specific input signal. This process not only aids in grasping the theoretical principles but also develops practical skills in system analysis, including convolution and response sketching. In this comprehensive guide, we will explore the methodology to determine $Y(t)$, techniques for sketching the response, and critical considerations for accurate analysis.

Understanding the Core Concepts

Before diving into the solution process, it is essential to understand the fundamental concepts involved:

Linear Time-Invariant (LTI) Systems

- Linearity: The system's response to a linear combination of inputs is the same linear combination of individual responses.
- Time-invariance: The system's properties do not change over time; its behavior is consistent regardless of when an input is applied.
- Impulse Response $h(t)$: The output of an LTI system when the input is an impulse $\delta(t)$. It completely characterizes the system's behavior.

System Response to Arbitrary Inputs

- The output $Y(t)$ can be obtained via the convolution integral:

$$
Y(t) = (x h)(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau
$$

- For causal systems, the limits of integration are often from 0 to t, simplifying calculations.

Significance of Sketching Responses

- Visualizing $Y(t)$ helps in understanding system behavior, such as transient and steady-state characteristics.
- Sketching aids in qualitative analysis, identifying features like peaks, zero-crossings, and asymptotic behavior.

Step-by-Step Process to Determine $Y(t)$

The primary approach involves applying the convolution integral or leveraging the Laplace or Fourier transforms, depending on the complexity of the functions involved.

1. Identify the Input Signal $x(t)$

- Clarify the form of the input signal such as step, impulse, sinusoid, or arbitrary waveform.
- Express the input mathematically for clarity, e.g., $x(t) = u(t)$ or $x(t) = \sin(\omega t)$.

2. Obtain the Impulse Response $h(t)$

- Given explicitly in the problem statement.
- If not directly given, it may need to be derived from system differential equations or transfer functions.

3. Set Up the Convolution Integral

- Write the convolution integral:

$$
Y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau
$$

- For causal systems, this simplifies to:

$$
Y(t) = \int_{0}^{t} x(\tau) h(t - \tau) d\tau
$$

4. Evaluate the Integral

- Break down the integral based on the support of $x(t)$ and $h(t)$.
- Use piecewise functions if $x(t)$ or $h(t)$ are defined in segments.

- Perform integration carefully, considering the signs and the domain of each function.

5. Analyze the Resulting Expression

- Simplify the expression to obtain a closed-form solution if possible.
- Examine the behavior for different time regions to understand transient and steady-state responses.

6. Sketch the Response $Y(t)$

- Use the obtained expression to plot $Y(t)$.
- Identify key features:
- Initial value $Y(0^-)$ and $Y(0^+)$
- Peak values and their timing
- Asymptotic behavior as $t \to \infty$ or $t \to -\infty$
- Zero-crossings and oscillations

Techniques for Simplifying and Evaluating the Response

Depending on the complexity, different mathematical techniques can be employed:

1. Using Laplace Transform

- Take the Laplace transform of $x(t)$ and $h(t)$:

$$
X(s) = \mathcal{L}\{x(t)\}, \quad H(s) = \mathcal{L}\{h(t)\}
$$

- Compute the product:

$$
Y(s) = X(s) \times H(s)
$$

- Find $Y(t)$ by inverse Laplace transform:

$$
Y(t) = \mathcal{L}^{-1}\{Y(s)\}
$$

- This approach simplifies convolution to multiplication in the s-domain.

2. Fourier Transform Approach

- Useful for signals and systems defined over the entire real line.
- Similar to Laplace but for steady-state and sinusoidal responses.

3. Piecewise Analysis

- Break down the input and impulse response into segments.
- Perform convolution over each segment and sum the results.

4. Graphical Convolution

- For qualitative understanding, perform time-domain graphing:
- Flip $h(t)$ around the vertical axis.
- Shift it by t.
- Multiply pointwise with $x(t)$.
- Integrate the area under the product to get $Y(t)$.

Sketching the Response $Y(t)$

Once the mathematical expression for $Y(t)$ is obtained, the next step is to sketch the response. This visual representation aids in comprehension and communication.

Key Steps in Sketching

- Determine critical points: times where the response changes behavior, peaks, or crosses zero.
- Identify asymptotic behavior: how $Y(t)$ behaves as $t \to \infty$ and $t \to -\infty$.
- Plot initial values: the response at $t=0^-$ and $t=0^+$.
- Sketch transient parts: peaks, overshoots, or undershoots.
- Draw steady-state segments: constant or oscillatory behavior as applicable.

Common Response Features

- Causality: Response begins at some initial time t_0 (usually zero for causal systems).
- Transient response: initial fluctuations before settling.
- Steady-state response: long-term behavior after transients decay.
- Oscillations: sinusoidal responses, their amplitude, and damping.
- Zero crossings: when the response crosses the horizontal axis.

Tools for Effective Sketching

- Use software like MATLAB, Wolfram Alpha, or graphing calculators for precise plots.
- Sketch by hand for conceptual understanding, marking key points and slopes.

Practical Example: Step-by-Step Illustration

To solidify the concepts, consider an example:

Given:
- Impulse response $h(t) = e^{-2t} u(t)$
- Input $x(t) = u(t)$ (unit step)

Solution:

1. Identify $x(t)$ and $h(t)$:
- $x(t) = u(t)$
- $h(t) = e^{-2t} u(t)$

2. Set up the convolution integral:

$$
Y(t) = \int_{0}^{t} u(\tau) \times e^{-2(t-\tau)} d\tau
$$

3. Evaluate the integral:

$$
Y(t) = \int_{0}^{t} e^{-2(t-\tau)} d\tau = e^{-2t} \int_{0}^{t} e^{2 \tau} d\tau
$$

$$
Y(t) = e^{-2t} \left[\frac{1}{2} e^{2 \tau} \right]_0^{t} = e^{-2t} \times \frac{1}{2} \left(e^{2t} - 1 \right)
$$

$$
Y(t) = \frac{1}{2} \left(1 - e^{-2t} \right)
$$

4. Sketch the response:
- As $t \to 0$, $Y(t) \to 0$.
- As $t \to \infty$, $Y(t) \to 1/2$.
- The response starts at zero and asymptotically approaches 0.5.

This example illustrates how convolution integrates the effects of the impulse response with the input to produce the output.
