In Each Of Problems 4 Through 9, Find The General Solution Of The Given Differential Equation. In Problems

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When tackling differential equations, a fundamental step is to determine their general solutions, which encompass all possible solutions to the given problem. These solutions often depend on arbitrary constants, reflecting the infinite set of functions satisfying the differential equation. Problems 4 through 9 typically involve various types of differential equations, including separable, linear, exact, and homogeneous equations. Understanding the methods to solve each type is crucial for systematically deriving the general solution. This article explores a step-by-step approach to solving these equations, emphasizing key techniques and common pitfalls.

Problem 4: Solving a Separable Differential Equation

Understanding Separable Differential Equations

A differential equation is called separable if it can be written in the form:

$$
\frac{dy}{dx} = f(x)g(y)
$$

which allows the variables to be separated on either side:

$$
\frac{1}{g(y)} dy = f(x) dx
$$

The main goal is to integrate both sides to find the general solution.

Methodology for Solving

1. Rewrite the differential equation in the separable form.
2. Separate the variables: all y's on one side, all x's on the other.
3. Integrate both sides: $\int \frac{1}{g(y)} dy = \int f(x) dx + C$.
4. Solve the resulting equation for y explicitly if possible, or leave it in implicit form.

Example

Suppose the differential equation is:

$$
\frac{dy}{dx} = y^2 \sin x
$$

Solution:

1. Rewrite as:

$$
\frac{1}{y^2} dy = \sin x dx
$$

2. Integrate both sides:

$$
\int y^{-2} dy = \int \sin x dx
$$

which yields:

$$
- y^{-1} = - \cos x + C
$$

3. Rearranged:

$$
\frac{1}{y} = \cos x + C'
$$

where $C' = -C$. The general solution:

$$
y(x) = \frac{1}{\cos x + C'}
$$

Problem 5: Solving a Linear Differential Equation

Understanding Linear Differential Equations

A first-order linear differential equation has the form:

$$
\frac{dy}{dx} + P(x) y = Q(x)
$$

where $P(x)$ and $Q(x)$ are continuous functions on an interval.

Methodology for Solution

1. Identify $P(x)$ and $Q(x)$.

2. Find the integrating factor (IF):

   $$
   \mu(x) = e^{\int P(x) dx}
   $$

3. Multiply the entire differential equation by $\mu(x)$, transforming it into:

   $$
   \frac{d}{dx} \left(\mu(x) y \right) = \mu(x) Q(x)
   $$

4. Integrate both sides:

   $$
   \mu(x) y = \int \mu(x) Q(x) dx + C
   $$

5. Finally, solve for y:

$$
y = \frac{1}{\mu(x)} \left(\int \mu(x) Q(x) dx + C \right)
$$

Example

Given:

$$
\frac{dy}{dx} + 2 y = e^{x}
$$

1. $P(x) = 2$, $Q(x) = e^{x}$.

2. Integrating factor:

$$
\mu(x) = e^{\int 2 dx} = e^{2x}
$$

3. Multiply the differential equation:

$$
e^{2x} \frac{dy}{dx} + 2 e^{2x} y = e^{3x}
$$

or

$$
\frac{d}{dx} \left(e^{2x} y \right) = e^{3x}
$$

4. Integrate:

$$
e^{2x} y = \int e^{3x} dx + C = \frac{1}{3} e^{3x} + C
$$

5. Solve for y:

$$
y = e^{-2x} \left(\frac{1}{3} e^{3x} + C \right) = \frac{1}{3} e^{x} + C e^{-2x}
$$

This is the general solution.

Problem 6: Solving an Exact Differential Equation

Understanding Exact Equations

An equation of the form:

$$
M(x, y) dx + N(x, y) dy = 0
$$

is exact if:

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

In such cases, there exists a potential function $\Psi(x, y)$ such that:

$$
\frac{\partial \Psi}{\partial x} = M(x, y), \quad \frac{\partial \Psi}{\partial y} = N(x, y)
$$

and the general solution is:

$$
\Psi(x, y) = C
$$

Methodology for Solving

1. Verify whether the differential equation is exact: check the equality of mixed partial derivatives.

2. If exact, find $\Psi(x, y)$ by integrating $M(x, y)$ with respect to x, treating y as constant.

3. Differentiate Ψ with respect to y and set equal to $N(x, y)$ to find any functions of x or y involved.

4. Write the implicit solution as $\Psi(x, y) = C$.

Example

Given:

$$
(2xy + y^2) dx + (x^2 + 2xy) dy = 0
$$

1. $M(x, y) = 2xy + y^2$, $N(x, y) = x^2 + 2xy$.

2. Check exactness:

$$
\frac{\partial M}{\partial y} = 2x + 2y
$$
$$
\frac{\partial N}{\partial x} = 2x + 2y
$$

They are equal, so the equation is exact.

3. Find $\Psi(x, y)$:

$$
\Psi_x = M = 2xy + y^2
$$

Integrate with respect to x:

$$
\Psi(x, y) = \int (2xy + y^2) dx = y \int 2x dx + y^2 \int dx = y x^2 + y^2 x + h(y)
$$

4. Differentiate Ψ with respect to y:

$$
\Psi_y = x^2 + 2 y x + h'(y)
$$

Set equal to $N(x, y) = x^2 + 2 xy$:

$$
x^2 + 2 y x + h'(y) = x^2 + 2 xy
$$

which simplifies to:

$$
h'(y) = 0
$$

\]

Thus, \(h(y) = \text{constant}\).

5. The implicit solution:

\[
\Psi(x, y) = y x^2 + y^2 x = C
\]

Problem 7: Homogeneous Differential Equations

Understanding Homogeneous Equations

A first-order differential equation is homogeneous if it can be expressed as:

\[
\frac{dy}{dx} = F\left(\frac{y}{x} \right)
\]

or equivalently, the right-hand side is a homogeneous function of degree zero.

Methodology for Solution

1. Make the substitution \(v = y/x\), which implies \(y = v x\).

2. Differentiate \(y = v x\):

   \[
   \frac{dy}{dx} = v + x \frac{dv}{dx}
   \]

3. Substitute into the original equation:

   \[
   v + x \frac{dv}{dx} = F(v)
   \]

4. Solve the resulting separable differential equation:

   \[

x \frac{dv}{dx} = F(v) - v
\]

which can be written as:

\[
\frac{dv}{F(v) - v} = \frac{dx}{x}
\]

5. Integrate both sides to find \(v\) in terms of \(x\), then back-substitute \(

Frequently Asked Questions

What is the general approach to solving differential equations in Problems 4 through 9?

The general approach involves identifying the type of differential equation (separable, linear, exact, etc.), applying the appropriate method to find the general solution, and including arbitrary constants to represent the family of solutions.

How do I recognize if a differential equation is separable in Problems 4 through 9?

A differential equation is separable if it can be written in the form dy/dx = f(x)g(y), allowing you to separate variables x and y and integrate both sides independently.

What is the significance of the arbitrary constant in the general solution of differential equations?

The arbitrary constant accounts for the family of solutions satisfying the differential equation, representing initial conditions or specific solution curves within the general solution.

How can I verify that my obtained general solution is correct for Problems 4 through 9?

You can differentiate the solution to see if it satisfies the original differential equation, or substitute it back into the equation to check for consistency.

Are there common pitfalls to avoid when solving problems 4 through 9?

Yes, common pitfalls include missing the arbitrary constant, incorrectly separating variables, forgetting to integrate, or mishandling initial conditions. Carefully following each step helps prevent these errors.

How do initial conditions affect the general solution in these problems?

Initial conditions allow you to determine the specific value of the arbitrary constant, leading to a particular solution tailored to given initial values.

Can all differential equations in Problems 4 through 9 be solved analytically?

Most linear, separable, or exact equations can be solved analytically, but some complex or non-standard equations may require numerical methods or special functions.

What are some tips for efficiently finding the general solution in Problems 4 through 9?

Identify the type of differential equation first, choose the appropriate method, carefully perform the integration, and double-check your work by substituting back into the original equation.

How do the solutions in Problems 4 through 9 relate to real-world applications?

General solutions to differential equations model various phenomena such as population growth, cooling processes, and oscillations, making understanding these solutions essential for practical problem-solving.

Additional Resources

Comprehensive Review of Solving Differential Equations in Problems 4 Through 9

Differential equations serve as a foundational pillar in mathematics, engineering, physics, and numerous applied sciences. They describe the relationships between functions and their derivatives, capturing the dynamics of systems ranging from simple growth processes to complex physical phenomena. When tackling problems 4 through 9, the primary objective is to find the general solution of each differential equation, which encapsulates all possible solutions by incorporating arbitrary constants. This review delves into the systematic approach, methodologies, and insights necessary for solving such equations, with an emphasis on clarity, depth, and practical understanding.

Understanding the Nature of Differential Equations

Before diving into solution strategies, it is crucial to understand the types of differential equations you might encounter:

- Ordinary Differential Equations (ODEs): Involve functions of a single independent variable and their derivatives.
- Partial Differential Equations (PDEs): Involve functions of multiple variables and their partial derivatives. (Not typically the focus here, but worth mentioning.)
- Order of Differential Equations: Determined by the highest derivative present.
- Linearity: Whether the equation is linear (the function and its derivatives appear linearly) or nonlinear.
- Homogeneity: Whether the equation can be expressed such that all terms are proportional with respect to the dependent variable or its derivatives.

For problems 4 through 9, the specific form of each differential equation will guide the choice of solution method.

General Strategies for Solving Differential Equations

1. Recognize the Type of Equation

The first step in solving any differential equation is classifying it. Common types include:

- Separable Equations: Can be written as $\frac{dy}{dx} = g(x)h(y)$.
- Linear Equations: Of the form $\frac{dy}{dx} + P(x)y = Q(x)$.
- Homogeneous Equations: Where the equation can be expressed in terms of ratios or scaled functions.
- Exact Equations: Satisfy the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ when written as $M(x,y)\,dx + N(x,y)\,dy=0$.

2. Simplify or Transform the Equation

Many equations become more manageable after substitution or transformation, such as:

- Substituting $v = y/x$ for homogeneous equations.
- Using integrating factors for linear equations.
- Variable substitution to convert nonlinear equations into linear forms.

3. Apply the Appropriate Solution Method

Based on the classification, choose a method:

- Separable Equation: Integrate both sides after separation.
- Linear Equation: Find an integrating factor $\mu(x) = e^{\int P(x) dx}$.
- Exact Equation: Find a potential function $\Psi(x,y)$.
- Homogeneous Equation: Use substitution to reduce to separable form.
- Bernoulli Equation: Transform into linear form via substitution $z = y^{1-n}$.

4. Integrate and Add the Constant

The general solution involves indefinite integrals and includes an arbitrary constant C, representing the family of solutions satisfying the differential equation.

In-Depth Analysis of Typical Problems 4 Through 9

While the specific equations are not provided, typical problems in this context involve familiar forms. Here's a detailed exploration of common differential equations encountered in such exercises.

Problem 4: Solving a Separable Differential Equation

Equation Form:
$$ \frac{dy}{dx} = g(x)h(y) $$

Approach:

- Step 1: Rewrite as $\frac{dy}{h(y)} = g(x) dx$.
- Step 2: Integrate both sides:
$$
\int \frac{dy}{h(y)} = \int g(x) dx + C
$$
- Step 3: Solve the integral, then solve for y if possible.

Example:
Suppose the problem is $\frac{dy}{dx} = xy$.

- Solution:
$$
\frac{dy}{y} = x dx
$$
Integrate:
$$
\ln|y| = \frac{x^2}{2} + C
$$
Exponentiate:
$$
y = \pm e^{C} e^{x^2/2} = C' e^{x^2/2}
$$

Key Takeaway:
Separable equations are often straightforward, relying on the ability to integrate both sides independently.

Problem 5: Linear Differential Equation

Equation Form:
$$ \frac{dy}{dx} + P(x) y = Q(x) $$

Solution Strategy:

- Step 1: Compute the integrating factor:
$$
\mu(x) = e^{\int P(x) dx}
$$

- Step 2: Multiply the entire differential equation by $\mu(x)$:

$$
\frac{d}{dx} [\mu(x) y] = \mu(x) Q(x)
$$

- Step 3: Integrate both sides:

$$
\mu(x) y = \int \mu(x) Q(x) dx + C
$$

- Step 4: Solve for y:

$$
y = \frac{1}{\mu(x)} \left(\int \mu(x) Q(x) dx + C \right)
$$

Example:
Equation: $\frac{dy}{dx} + 2 y = e^{3x}$

- Compute $\mu(x) = e^{\int 2 dx} = e^{2x}$.
- Multiply through:

$$
e^{2x}\frac{dy}{dx} + 2 e^{2x} y = e^{5x}
$$

- Recognize the left as derivative:

$$
\frac{d}{dx} [e^{2x} y] = e^{5x}
$$

- Integrate:

$$
e^{2x} y = \int e^{5x} dx + C = \frac{e^{5x}}{5} + C
$$

- Solve for y:

$$
y = e^{-2x} \left(\frac{e^{5x}}{5} + C \right) = \frac{e^{3x}}{5} + C e^{-2x}
$$

Key Points:
Mastering integrating factors and recognizing derivatives of products are essential skills for linear equations.

Problem 6: Homogeneous Differential Equation

Equation Form:

$$ M(x,y) dx + N(x,y) dy = 0 $$

where the functions are homogeneous of the same degree.

Method:

- Step 1: Check homogeneity by verifying that $M(tx, ty) = t^n M(x,y)$

M and similarly for N.
- Step 2: Use substitution $v = y/x$, which implies $y = vx$, $dy = v dx + x dv$.
- Step 3: Rewrite the equation in terms of v and x, leading to a separable form:
$$
\frac{dy}{dx} = v + x \frac{dv}{dx}
$$
- Step 4: Solve for v and then back-substitute to find y.

Example:
Equation: $y' = \frac{y}{x} + \frac{x}{y}$.

- Recognize the form involves homogeneous functions.
- Substituting $y = vx$:
$$
y' = v + x \frac{dv}{dx}
$$
- Rewrite the original:
$$
v + x \frac{dv}{dx} = v + \frac{x}{v x} = v + \frac{1}{v}
$$
- Simplify:
$$
x \frac{dv}{dx} = \frac{1}{v}
$$
- Separate:
$$
v dv = \frac{dx}{x}
$$
- Integrate:
$$
\frac{v^2}{2} = \ln|x| + C
$$
- Back-substitute $v = y/x$:
$$
\frac{(y/x)^2}{2} = \ln|x| + C
$$
- Final implicit solution:
$$
y^2 = 2 x^2 (\ln|x| + C)
$$

Significance:
Homogeneous equations often simplify with substitution, revealing solutions that are not immediately obvious.

Problem 7: Bernoulli Differential Equation

Equation Form:
\[\frac{dy}{dx} + P(x) y = Q(x) y^n \]

where \(n \neq 0,1 \).

Solution Technique:

- Step 1: Divide through by \(y^n \) to obtain:
\[
y^{-n} \frac{dy}{dx} + P(x) y^{1 - n} = Q(x)

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