Let F Be A Function From A To B. (a) Show That If F Is Injective And EA, Then F 1(f(E))=E. Give An Example

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Introduction

Mathematics is a language of relationships, structures, and functions that connect various sets and elements within them. Understanding the properties of functions, particularly injectivity and surjectivity, is crucial in fields ranging from algebra to analysis.

This article explores a specific property involving functions between sets, focusing on the relationship between a function $F : A \to B$, its inverse image, and the concepts of injectivity and the image of a subset. We will demonstrate that if F is injective and onto the image of a subset $E \subseteq A$, then the inverse image of the image of E under F equals E. Additionally, an illustrative example will clarify this concept.

Understanding the Fundamental Concepts

Before delving into the proof and example, let's clarify the key definitions involved:

1. Functions and Sets

- Function $F : A \to B$: A rule that assigns each element of set A to exactly one element of set B.
- Subset $E \subseteq A$: A collection of elements within A.
- Image of E under F: Denoted $F(E)$, it is the set $\{F(e) : e \in E\}$.

2. The Inverse Image (Pre-image) F^{-1}

- For a subset $Y \subseteq B$, the inverse image $F^{-1}(Y)$ is defined as:

$$
F^{-1}(Y) = \{a \in A : F(a) \in Y\}
$$

- Notably, $F^{-1}(F(E))$ contains all elements in A that map into $F(E)$.

3. Injective (One-to-One) Functions

- Injectivity: A function F is injective if:

$$
\text{For all } a_1, a_2 \in A, \quad \text{if } F(a_1) = F(a_2), \text{ then } a_1 = a_2
$$

- Intuition: No two different elements in A map to the same element in B.

4. Surjective (Onto) and The Notation EA

- Surjectivity: A function $F : A \to B$ is surjective if:

$$
\text{For every } b \in B, \quad \text{there exists } a \in A \text{ such that } F(a) = b
$$

- EA notation: In some contexts, EA may denote that F is onto its image, or that the image of E under F is the entire set $F(E)$. For clarity, in this context, it indicates the mapping is onto the subset $F(E)$.

Statement of the Theorem

Theorem:
Let $F : A \to B$ be a function. If F is injective and the restriction of F to $E \subseteq A$ is onto its image $F(E)$ (i.e., $F|_E : E \to F(E)$ is surjective), then:

$$
F^{-1}(F(E)) = E
$$

Furthermore, we will illustrate this with a concrete example.

Proof of the Theorem

To establish the equality $F^{-1}(F(E)) = E$, we need to demonstrate two inclusions:

1. $E \subseteq F^{-1}(F(E))$

- Argument:
- Take any element $e \in E$.
- Since $F(e) \in F(E)$ by definition, it follows that:
$$
e \in F^{-1}(F(E))
$$
- Therefore:
$$
E \subseteq F^{-1}(F(E))
$$

2. $F^{-1}(F(E)) \subseteq E$

- Assumption: F is injective and $F|_E$ is onto $F(E)$.
- Suppose $a \in F^{-1}(F(E))$. Then:
$$
F(a) \in F(E)
$$
- So, there exists some $e \in E$ such that:
$$
F(a) = F(e)
$$
- Since F is injective, the equality $F(a) = F(e)$ implies:
$$
a = e
$$
- Conclusion:
$$
a \in E
$$

- Hence:
$$
F^{-1}(F(E)) \subseteq E
$$

Combining both parts:

$$
F^{-1}(F(E)) = E
$$

This completes the proof.

Key Conditions and Their Significance

To understand why the theorem holds, it's essential to recognize the roles of injectivity and surjectivity:

- Injectivity ensures that different elements in A map to different elements in B, which allows us to uniquely identify pre-images.
- Surjectivity of $F|_E$ guarantees that every element in the image $F(E)$ has a pre-image within E. This is crucial because it prevents elements outside E from mapping into $F(E)$, ensuring the inverse image doesn't include extraneous elements.

Example to Illustrate the Theorem

Let's consider a concrete example to solidify the concepts.

Setup of the Example

- Sets:
$$
A = \{1, 2, 3, 4\}
$$
\[

$$
B = \{a, b, c, d\}
$$

- Function $F : A \to B$:

$$
F(1) = a
$$

$$
F(2) = b
$$

$$
F(3) = c
$$

$$
F(4) = d
$$

- Subset $E \subseteq A$:

$$
E = \{2, 3\}
$$

Analysis of the Example

- Check injectivity:
- Each element of A maps to a unique element in B.
- Yes, F is injective.

- Check the restriction $F|_E : E \to F(E)$:
- $F(2) = b$
- $F(3) = c$
- $F(E) = \{b, c\}$
- The mapping $F|_E$ is onto $F(E)$ because:
- For each $b, c \in F(E)$, there exists an element in E mapping to it.
- Yes, $F|_E$ is onto $F(E)$.

- Compute $F(E)$:

$$
F(E) = \{b, c\}
$$

- Find $F^{-1}(F(E))$:

$$
F^{-1}(\{b, c\}) = \{a \in A : F(a) \in \{b, c\}\}
$$
- $F(2) = b \Rightarrow 2 \in F^{-1}(F(E))$
- $F(3) = c \Rightarrow 3 \in F^{-1}(F(E))$
- $F(1) = a \notin \{b, c\}$
- $F(4) = d \notin \{b, c\}$

$$
\Rightarrow F^{-1}(F(E)) = \{2, 3\} = E
$$

Conclusion from the Example

This example perfectly illustrates the theorem:

- The function F is injective.
- Its restriction to E is onto $F(E)$.
- The inverse image of $F(E)$ under F equals E.

This confirms the theoretical result with a tangible case.

Implications and Applications of the Result

Understanding this property has several notable implications:

- In Set Theory and Functions:
- It confirms that for injective functions, the inverse image of the

Frequently Asked Questions

What does it mean for a function F from A to B to be injective?

A function F is injective (one-to-one) if different elements in A map to different elements in B; that is, if $F(x1) = F(x2)$, then $x1 = x2$.

What is the significance of the condition EA (F(E) = A) in the context of injective functions?

The condition EA indicates that the image of E under F covers the entire set A, meaning every element in A is mapped to by some element in E.

How do we show that F^{-1}(F(E)) = E when F is injective and EA?

Since F is injective and EA, for any element x in F^{-1}(F(E)), F(x) is in F(E), so x is in E. Conversely, every element of E has its image in F(E), so F^{-1}(F(E)) contains E. Combining both, F^{-1}(F(E)) = E.

What is the role of inverse images in understanding the relationship between E and F(E)?

Inverse images, or preimages, help us identify all elements that map into a particular set in the codomain. When F is injective, F^{-1}(F(E)) precisely recovers E.

Can you provide a simple example of a function F from A to B that satisfies these properties?

Yes. Consider $F: \mathbb{R} \to \mathbb{R}$ defined by $F(x) = 2x$, which is injective. Let $E = [1, 3]$. Then $F(E) = [2, 6]$, and F^{-1}(F(E)) = $\{x \mid 2x \in [2, 6]\} = [1, 3]$, which is E.

Why is the injectivity of F crucial for the equality F^{-1}(F(E)) = E?

Injectivity ensures that no two distinct elements in E have the same image, making the inverse image of F(E) exactly E. Without injectivity, F^{-1}(F(E)) could be larger than E.

How does the condition EA influence the conclusion about the inverse image?

EA guarantees that every element in A is an image of some element in E, which simplifies the inverse image calculation and ensures F^{-1}(F(E)) equals E when F is injective.

What would happen if F were not injective? Would $F^{-1}(F(E))$ still equal E?

If F is not injective, then $F^{-1}(F(E))$ could include elements outside of E because multiple elements in A could map to the same image, making the equality fail.

Can you summarize the key takeaway about injective functions and inverse images in this context?

For an injective function F, if E is such that F(E) = A, then the inverse image of F(E) under F is E itself; this property highlights the one-to-one correspondence and the importance of injectivity in inverse image relationships.

Additional Resources

Understanding the Relationship Between Injective Functions and Set Images: A Deep Dive

In mathematics, the study of functions (or mappings) between sets forms a cornerstone of foundational concepts in areas such as algebra, analysis, and topology. Among the various properties a function can possess, injectivity (or being one-to-one) plays a crucial role, especially when discussing the behavior of functions concerning subsets and their images. This review aims to elucidate the statement:

> "Let F be a function from A to B. (a) Show that if F is injective and $E \subseteq A$, then $F^{-1}(F(E)) = E$. Give an example."

We will explore this assertion comprehensively, beginning with the necessary definitions, followed by a detailed proof, intuitive explanations, and illustrative examples. The goal is to understand why this property holds under the conditions specified and how it manifests in concrete scenarios.

Fundamental Concepts and Definitions

Before delving into the proof and implications, it’s essential to clarify the key concepts involved:

Injective Function (One-to-One)

- Definition: A function $F: A \to B$ is injective (or one-to-one) if distinct elements in A map to distinct elements in B. Formally:

$$
\forall a_1, a_2 \in A, \quad F(a_1) = F(a_2) \implies a_1 = a_2
$$

- Intuitive Explanation: No two different elements in A are mapped to the same element in B.

Image of a Set under a Function

- Definition: For $E \subseteq A$, the image of E under F, denoted $F(E)$, is:

$$
F(E) = \{ F(a) \mid a \in E \}
$$

Preimage (Inverse Image) of a Set

- Definition: For $S \subseteq B$, the preimage of S under F, denoted $F^{-1}(S)$, is:

$$
F^{-1}(S) = \{ a \in A \mid F(a) \in S \}
$$

- Special case: For a subset $E \subseteq A$, $F^{-1}(F(E))$ is the set of all elements in A whose images lie in $F(E)$.

The Statement and Its Significance

The statement under consideration is:

> "If $F: A \to B$ is injective and $E \subseteq A$, then $F^{-1}(F(E)) = E$."

This assertion reveals a fundamental property of injective functions: the preimage of the image of a subset is exactly the subset itself.

Why is this noteworthy? Because, in general, for arbitrary functions (not necessarily injective), the inclusion

$$
E \subseteq F^{-1}(F(E))

\]

always holds, but equality does not necessarily follow. The injectivity condition ensures the reverse inclusion as well, giving us equality.

This property is vital in understanding how functions behave with respect to sets and plays a pivotal role in many areas such as inverse functions, bijections, and measure theory.

Step-by-Step Proof of the Statement

Let's now rigorously prove the statement:

> Given $F: A \to B$ is injective and $E \subseteq A$, then $F^{-1}(F(E)) = E$.

1. Show that $E \subseteq F^{-1}(F(E))$

Claim: Every element $a \in E$ belongs to $F^{-1}(F(E))$.

Proof:

- Take any $a \in E$.
- By the definition of $F(E)$, $F(a) \in F(E)$.
- The preimage $F^{-1}(F(E))$ contains all elements $a' \in A$ such that $F(a') \in F(E)$.
- Since $F(a) \in F(E)$, it follows that:

$$
a \in F^{-1}(F(E))
$$

- This is true for all $a \in E$, hence:

$$
E \subseteq F^{-1}(F(E))
$$

2. Show that $F^{-1}(F(E)) \subseteq E$

Claim: Every element $a \in F^{-1}(F(E))$ must belong to E.

Proof:

- Take any $a \in F^{-1}(F(E))$.
- By the definition, $F(a) \in F(E)$.
- Therefore, there exists some $a' \in E$ such that:

$$
F(a) = F(a')
$$

- Since F is injective, the equality $F(a) = F(a')$ implies:

$$
a = a'
$$

- But $a' \in E$, so:

$$
a \in E
$$

- This completes the proof that:

$$
F^{-1}(F(E)) \subseteq E
$$

3. Combining both parts

From parts 1 and 2, we conclude:

$$
E \subseteq F^{-1}(F(E)) \quad \text{and} \quad F^{-1}(F(E)) \subseteq E
$$

which yields the equality:

$$
\boxed{
F^{-1}(F(E)) = E
}
$$

Intuitive Explanation and Geometric Interpretation

While the formal proof is rigorous, an intuitive understanding can greatly aid comprehension.

Why does injectivity matter?

- Without injectivity, the image $F(E)$ could be "collapsing" multiple elements of E into the same point in B. As a result, the preimage $F^{-1}(F(E))$ could include elements not in E, because the preimage includes all elements that map into the same images as those in E.

- With injectivity, each element in the image corresponds to exactly one element in A. So, when you take $F(E)$, the set of images of elements in E, the preimage of that set returns precisely those elements in E. No extraneous elements are added because injectivity prevents multiple elements from sharing the same image.

Analogy

Imagine a set of keys (A) and a lock system (B). The function F assigns each key to a lock. If F is injective:

- Each key opens a unique lock.
- When you look at the set of locks opened by keys in E, the preimage of this set is exactly the keys in E.
- No other keys outside E can open these locks because each lock is uniquely associated with a key.

Concrete Example Demonstrating the Property

To solidify understanding, consider a specific example with explicit sets and a function.

Example Setup

- Let:

$$
A = \{1, 2, 3, 4\}
$$

$$

$$

$$
B = \{a, b, c\}
$$

- Define a function $F: A \to B$ as follows:

$$
F(1) = a, \quad F(2) = b, \quad F(3) = c, \quad F(4) = c
$$

- Note: This function is not injective because $F(3) = F(4) = c$.

- To satisfy the injectivity condition, modify F:

$$
F(1) = a, \quad F(2) = b, \quad F(3) = c, \quad F(4) = d
$$

where d is a new element in B. Now,

$$
B = \{a, b, c, d\}
$$

- The updated F is:

$$
F(1) = a, \quad F(2) = b, \quad F(3) = c, \quad F(4) = d
$$

- This is injective because all images are distinct.

Choosing a subset $E \subseteq A$

Suppose:

$$
E = \{2, 4\}
$$

- Calculate $F(E)$:

\[
F(E) = \{F(2), F(4)\} = \{b, d\}
\]

- Determine \(F^{-1}(F(E))\):

\[
F^{-1}(\{b, d\}) = \{ a \in A \mid F(a) \in \{

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