(20 Pts) Q2) Determine The Fourier Transform Of The Following Signals (Show Your Work, Don't Use FT Table):

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Introduction

Fourier Transform (FT) is a fundamental mathematical tool in signal processing, electrical engineering, and applied physics. It allows us to analyze signals in the frequency domain, revealing the spectrum of frequencies that compose a time-domain signal. Understanding how to manually compute the Fourier Transform without relying solely on standard tables is crucial for developing a deep grasp of signal behavior and the underlying mathematics.

This article provides a comprehensive, step-by-step guide to determine the Fourier Transform of various signals. We will focus on the methodology, detailed calculations, and practical insights to ensure clarity and mastery in signal analysis.

Fundamentals of Fourier Transform

Before diving into specific signals, let's review the basic definition and properties of the Fourier Transform.

Definition of Fourier Transform

For a time-domain signal $x(t)$, its Fourier Transform $X(f)$ is given by:

$$
X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt
$$

where:
- t is the time variable,
- f is the frequency variable,
- j is the imaginary unit.

Key Properties

Understanding certain properties simplifies the transformation of common signals:
- Linearity
- Time shifting
- Frequency shifting
- Scaling

- Convolution

Approach to Computing Fourier Transforms Without FT Tables

When tasked with finding the Fourier Transform without tables, apply these general steps:

1. Identify the signal type: Recognize if it's a basic waveform, exponential, shifted, etc.
2. Express the signal mathematically: Write the signal in a form suitable for integration.
3. Use known integral formulas: Recall basic integral results for exponentials, sines, cosines, delta functions, etc.
4. Apply properties if necessary: For shifted or scaled signals, use the properties of Fourier Transforms.
5. Simplify the integral: Perform the integral step-by-step, considering convergence conditions.
6. Interpret the result: Express the Fourier Transform in a simplified, standard form.

Example 1: Fourier Transform of a Rectangular Pulse

Signal Definition

Consider a rectangular pulse:

$$
x(t) =
\begin{cases}
1, & |t| \leq \frac{T}{2} \\
0, & |t| > \frac{T}{2}
\end{cases}
$$

Step-by-Step Calculation

Step 1: Write the Fourier Transform:

$$
X(f) = \int_{-\infty}^{\infty} x(t) e^{-j 2 \pi f t} dt
$$

Since $x(t)$ is non-zero only between $-T/2$ and $T/2$:

$$
X(f) = \int_{-T/2}^{T/2} e^{-j 2 \pi f t} dt
$$

Step 2: Perform the integral:

$$
X(f) = \left[\frac{e^{-j 2 \pi f t}}{-j 2 \pi f} \right]_{-T/2}^{T/2}
$$

\]

\[
X(f) = \frac{1}{-j 2 \pi f} \left(e^{-j 2 \pi f (T/2)} - e^{j 2 \pi f (T/2)} \right)
\]

Step 3: Simplify numerator:

\[
e^{-j \pi f T} - e^{j \pi f T} = -2j \sin (\pi f T)
\]

Step 4: Final expression:

\[
X(f) = \frac{1}{-j 2 \pi f} \times (-2j) \sin (\pi f T) = \frac{\sin (\pi f T)}{\pi f}
\]

Result:

\[
\boxed{
X(f) = T \cdot \operatorname{sinc} (\pi f T)
}
\]

where \(\operatorname{sinc}(x) = \frac{\sin x}{x} \).

Example 2: Fourier Transform of a Discrete-Time Impulse

Signal Definition

The Dirac delta function:

\[
x(t) = \delta(t - t_0)
\]

Step-by-Step Calculation

Step 1: Write the Fourier Transform:

\[
X(f) = \int_{-\infty}^\infty \delta(t - t_0) e^{-j 2 \pi f t} dt
\]

Step 2: Use the sifting property of delta:

\[

X(f) = e^{-j 2 \pi f t_0}
\]

Result:

\[
\boxed{
X(f) = e^{-j 2 \pi f t_0}
}
\]

This result shows that a time shift introduces a linear phase term in the frequency domain.

Example 3: Fourier Transform of an Exponential Signal

Signal Definition

Consider the causal exponential:

\[
x(t) = e^{a t} u(t)
\]

where \(u(t) \) is the unit step function, and \(a \) is a complex constant with \(\operatorname{Re}(a) < 0 \) for convergence.

Step-by-Step Calculation

Step 1: Write the Fourier Transform:

\[
X(f) = \int_{0}^{\infty} e^{a t} e^{-j 2 \pi f t} dt
\]

Step 2: Combine exponents:

\[
X(f) = \int_{0}^{\infty} e^{(a - j 2 \pi f) t} dt
\]

Step 3: Integrate (assuming convergence):

\[
X(f) = \left[\frac{e^{(a - j 2 \pi f) t}}{a - j 2 \pi f} \right]_0^{\infty}
\]

Since \(\operatorname{Re}(a) < 0 \), the exponential decays:

\[

$$
X(f) = \frac{0 - 1}{a - j 2 \pi f} = \frac{1}{j 2 \pi f - a}
$$

Result:

$$
\boxed{
X(f) = \frac{1}{j 2 \pi f - a}
}
$$

Additional Examples and Common Signals

1. Sinusoidal Signal

For a sinusoid $x(t) = \cos(2 \pi f_0 t)$, the Fourier Transform yields delta functions at $\pm f_0$:

$$
X(f) = \frac{1}{2} \left[\delta(f - f_0) + \delta(f + f_0) \right]
$$

2. Shifted and Scaled Signals

Using properties:

- Time shift: $x(t - t_0) \leftrightarrow e^{-j 2 \pi f t_0} X(f)$
- Scaling: $x(a t) \leftrightarrow \frac{1}{|a|} X \left(\frac{f}{a} \right)$

Practical Tips for Manual Fourier Transform Calculation

- Use substitution methods to simplify integrals.
- Recognize standard integral forms (e.g., integral of exponential functions).
- Carefully consider convergence conditions.
- Use properties of the Fourier Transform to handle shifts and scaling efficiently.
- Confirm the physical relevance of the result (e.g., causality, energy spectrum).

Conclusion

Calculating the Fourier Transform without relying solely on tables enhances conceptual understanding and analytical skills. By systematically applying the integral definition, properties, and known integral results, one can derive the spectral content of various signals. Practice with different functions—including pulses, exponentials, and shifted signals—solidifies the ability to analyze signals in the frequency domain.

Mastery of this process is essential for engineers and scientists working in signal processing, communications, and related fields. The key is to approach each problem with a clear strategy, patience, and attention to mathematical detail, leading to accurate and insightful frequency domain representations.

References

- Oppenheim, A.V., Willsky, A.S., & Nawab, S.H. (1997). Signals and Systems. Prentice Hall.
- Bracewell, R. N. (2000). The Fourier Transform and Its Applications. McGraw-Hill.
- Papoulis, A. (1962). The Fourier Integral and Its Applications. McGraw-Hill.

This detailed guide aims to equip you with the skills necessary to compute Fourier Transforms of various signals manually, without the reliance on tables, fostering a deeper understanding of their mathematical foundations.

Frequently Asked Questions

How do you find the Fourier Transform of a rectangular pulse signal without using the FT table?

To find the Fourier Transform of a rectangular pulse, represent the pulse as a rectangular function, then apply the definition of the Fourier Transform: $F(\omega) = \int x(t) e^{-j\omega t} dt$ over the pulse's duration. For a pulse of amplitude A, width T, centered at t=0: $F(\omega) = A \int_{-T/2}^{T/2} e^{-j\omega t} dt = A [(e^{-j\omega t}) / (-j\omega)]_{-T/2}^{T/2} = A (\sin(\omega T/2) / (\omega/2)) = A T \text{sinc}(\omega T/2)$.

What is the Fourier Transform of a shifted delta function $\delta(t - t_0)$, and how is it derived without using the FT table?

The Fourier Transform of $\delta(t - t_0)$ is $e^{-j\omega t_0}$. Derivation: $F(\omega) = \int \delta(t - t_0) e^{-j\omega t} dt$. Since the delta function picks out the value at $t = t_0$, $F(\omega) = e^{-j\omega t_0}$. This result holds regardless of using a table, by directly applying the definition of the delta function.

How do you compute the Fourier Transform of an exponential decay signal $x(t) = e^{-a t} u(t)$ (where u(t) is the unit step function) without using FT tables?

Using the definition, $F(\omega) = \int_{0}^{\infty} e^{-a t} e^{-j\omega t} dt = \int_{0}^{\infty} e^{-(a + j\omega) t} dt$. Evaluating the integral gives $F(\omega) = 1 / (a + j\omega)$, assuming $\text{Re}(a) > 0$ for convergence. This derivation involves integrating a simple exponential function directly.

Describe the process to find the Fourier Transform of a cosine

signal $\cos(\omega_0 t)$ without referring to FT tables.

Express $\cos(\omega_0 t)$ as $(e^{j\omega_0 t} + e^{-j\omega_0 t}) / 2$. Then, find the Fourier Transform of each exponential term: the FT of $e^{j\omega_0 t}$ is $2\pi \delta(\omega - \omega_0)$, and that of $e^{-j\omega_0 t}$ is $2\pi \delta(\omega + \omega_0)$. Therefore, the FT of $\cos(\omega_0 t)$ is $\pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$. This approach avoids tables by using exponential definitions.

Additional Resources

Fourier Transform of Signals: A Comprehensive Analysis

Understanding the Fourier Transform (FT) is fundamental in signal processing, communications, and many engineering disciplines. It provides a powerful tool to analyze signals in the frequency domain, revealing the spectral components that constitute a time-domain signal. In this detailed review, we will methodically explore how to determine the Fourier Transform of various signals, emphasizing the process over rote memorization of FT tables. Our focus will be on demonstrating the derivation step-by-step, with illustrative examples, and discussing key properties and techniques involved.

Introduction to Fourier Transform

The Fourier Transform converts a time-domain signal $x(t)$ into its frequency-domain representation $X(f)$. It is defined as:

$$
X(f) = \int_{-\infty}^{\infty} x(t) e^{-j 2 \pi f t} dt
$$

Similarly, the inverse Fourier Transform reconstructs the time signal:

$$
x(t) = \int_{-\infty}^{\infty} X(f) e^{j 2 \pi f t} df
$$

The Fourier Transform exists for a wide class of signals, especially those that are integrable or square-integrable, and it reveals spectral content, phase information, and enables filtering, modulation, and system analysis.

Approach to Deriving Fourier Transforms Without Using Tables

When asked to determine the Fourier Transform of signals without consulting tables, we rely on the core properties, basic transforms, and algebraic manipulations. The main steps include:

1. Identify the signal: Recognize the form of the signal—exponential, sinusoid, rectangular pulse, etc.
2. Decompose complex signals: Use properties like linearity, time-shifting, frequency-shifting, scaling, and modulation.
3. Recall fundamental transforms:
- The transform of $e^{a t} u(t)$
- The transform of delta functions $\delta(t)$
- The Fourier transform of sinusoidal functions
4. Apply properties: Use linearity, time scaling, and shift properties to relate the given signal to known transforms.
5. Perform integrations carefully: When necessary, compute the integral directly, especially for piecewise functions or signals involving exponentials and sinusoids.
6. Simplify the result: Express the Fourier Transform in closed-form, often involving rational functions, sinc functions, or delta functions.

Example Signals and Their Fourier Transforms

Let’s consider some common signals and derive their Fourier Transforms step-by-step to illustrate the process.

1. Fourier Transform of a Rectangular Pulse

Signal:

$$
x(t) = \begin{cases}
A, & |t| \leq T/2 \\
0, & \text{otherwise}
\end{cases}
$$

Derivation:

- Step 1: Write the FT integral:

$$
X(f) = \int_{-\infty}^{\infty} x(t) e^{-j 2 \pi f t} dt
$$

- Step 2: Since $x(t)$ is non-zero only between $-T/2$ and $T/2$:

$$
X(f) = A \int_{-T/2}^{T/2} e^{-j 2 \pi f t} dt
$$

- Step 3: Evaluate the integral:

$$
X(f) = A \left[\frac{e^{-j 2 \pi f t}}{-j 2 \pi f} \right]_{t=-T/2}^{T/2} = A \left(\frac{e^{-j \pi f T} - e^{j \pi f T}}{-j 2 \pi f} \right)
$$

- Step 4: Recognize the numerator as a sine function:

$$
e^{-j \pi f T} - e^{j \pi f T} = -2j \sin(\pi f T)
$$

- Step 5: Substitute back:

$$
X(f) = A \times \frac{-2j \sin(\pi f T)}{-j 2 \pi f} = A \times \frac{2j \sin(\pi f T)}{j 2 \pi f}
$$

- Step 6: Simplify:

$$
X(f) = A \times \frac{\sin(\pi f T)}{\pi f}
$$

Result:

$$
\boxed{
X(f) = A T \, \text{sinc}(f T)
}
$$

where $\text{sinc}(x) = \frac{\sin \pi x}{\pi x}$.

Note: The derivation shows how the rectangular pulse's spectral content is a sinc function, illustrating the time-frequency duality.

---

2. Fourier Transform of an Exponential Signal

Signal:

$$
x(t) = e^{a t} u(t)
$$

where $u(t)$ is the unit step function, and a is a real constant.

Derivation:

- Step 1: Write the integral:

$$
X(f) = \int_{-\infty}^{\infty} e^{a t} u(t) e^{-j 2 \pi f t} dt
$$

- Step 2: Since $u(t) = 0$ for $t<0$:

$$
X(f) = \int_{0}^{\infty} e^{a t} e^{-j 2 \pi f t} dt = \int_{0}^{\infty} e^{(a - j 2 \pi f) t} dt
$$

- Step 3: Evaluate the integral, assuming convergence:

$$
X(f) = \left[\frac{e^{(a - j 2 \pi f) t}}{a - j 2 \pi f} \right]_0^{\infty}
$$

- Step 4: For the integral to converge, the real part of the exponent must be negative:

$$
\text{Re}(a) < 0
$$

- Step 5: Under this condition, the exponential at infinity tends to zero:

$$
X(f) = \frac{0 - 1}{a - j 2 \pi f} = \frac{1}{j 2 \pi f - a}
$$

Result:

$$
\boxed{
X(f) = \frac{1}{j 2 \pi f - a}
}
$$

This result indicates that the Fourier transform of a causal exponential is a simple rational function with a pole at $f = \frac{a}{j 2 \pi}$.

---

3. Fourier Transform of a Sinusoid

Signal:

$$
x(t) = \cos(2 \pi f_0 t)
$$

Derivation:

- Step 1: Express the cosine as exponential:

$$
\cos(2 \pi f_0 t) = \frac{1}{2} \left(e^{j 2 \pi f_0 t} + e^{-j 2 \pi f_0 t} \right)
$$

- Step 2: The FT of $e^{j 2 \pi f_0 t}$ is $\delta(f - f_0)$, similarly for $e^{-j 2 \pi f_0 t}$:

$$
\mathcal{F}\{ e^{j 2 \pi f_0 t} \} = \delta(f - f_0)
$$

$$
\mathcal{F}\{ e^{-j 2 \pi f_0 t} \} = \delta(f + f_0)
$$

- Step 3: Applying linearity:

$$
X(f) = \frac{1}{2} \left(\delta(f - f_0) + \delta(f + f_0) \right)
$$

Result:

$$
\boxed{
X(f) = \frac{1}{2} \left[\delta(f - f_0) + \delta(f + f_0)\right]
}
$$

This shows that a sinusoid in time corresponds to two spectral spikes at $\pm f_0$.

Techniques for Deriving Fourier Transforms of Complex Signals

While the above examples are straightforward, many practical signals are more complex. Here are techniques and properties useful for deriving their FT:

1. Linearity

$$
\mathcal{F}\{a x(t) + b y(t)\} = a X(f) + b Y(f)
$$

This allows breaking down signals into simpler components.

2. Time Shifting

$$
\mathcal{F}\{x(t - t_0)\} = X(f) e^{-j 2 \pi f t_0}
$$

Useful when signals are delayed.

3. Scaling in Time

$$
\mathcal{F}\{x(a t)\} = \frac{1}{|a|} X\left(\frac{f}{a}\right)
$$

Enables analyzing signals with time compression or expansion.

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