4. Let {u, V} Be Linearly Independent. (a) Determine If {u, V, 2+ V} Is Linearly Independent. (b) Determine

Introduction

Linear independence is a fundamental concept in linear algebra that plays a crucial role in understanding the structure of vector spaces. When working with vectors, determining whether a set is linearly independent or dependent impacts the basis, dimension, and the solutions to associated linear systems. This article explores a specific problem involving the linear independence of a set of vectors, focusing on two key parts: (a) assessing whether the set {u, V, 2 + V} remains linearly independent given that {u, V} are initially independent, and (b) analyzing a subsequent related question. We will systematically dissect the problem, providing detailed explanations and step-by-step reasoning to clarify the concepts involved.

Understanding the Problem Statement

Given Data and Definitions

- Vectors **u** and **V** are linearly independent.
- The set in question for part (a) is *{u, V, 2 + V}*.
- The problem asks whether adding the vector *2 + V* to the set with *u* and *V* affects linear independence.

Note: The notation *2 + V* suggests a vector that is the sum of a scalar (2) and the vector *V*. Typically, in linear algebra, vectors are not added to scalars directly. To interpret this correctly, we need to clarify whether *2 + V* is a vector obtained by adding 2 times some basis vector or scalar multiple, or if it's an expression involving scalar addition. For the purpose of this problem, we assume that *2 + V* is a vector that can be written as *2e + V* where *e* is a scalar or basis vector, or that the problem intends to consider the vector *V + 2w* for some vector *w*. Alternatively, if the notation indicates scalar addition, it might be a typo or shorthand. For clarity, we'll proceed under the assumption that *2 + V* is intended to be a vector expressed as *V + c* for some scalar *c*, i.e., a vector obtained by adding a scalar multiple of some fixed vector or scalar to *V*.

Part (a): Determining If {u, V, 2 + V} Is Linearly

Independent

Step 1: Recall the Definition of Linear Independence

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ is linearly independent if the only solution to the linear combination equation

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$$

is when all scalars αi are zero.

In our case, for the set $\{u, V, 2 + V\}$, linear independence holds if the only solution to

$$\alpha u + \beta V + \gamma (2 + V) = 0$$

is when $\alpha = \beta = \gamma = 0$.

Step 2: Express the Linear Combination and Simplify

Rewrite the linear combination:

$$\alpha u + \beta V + \gamma (2 + V) = 0$$

which expands to:

$$\alpha u + \beta V + \gamma \ 2 + \gamma V = 0$$

or equivalently:

$$\alpha u + (\beta + \gamma) V + 2\gamma = 0$$

Note: The term 2γ is a scalar multiple of the scalar 2, which is not a vector. This suggests that the initial assumption about $2 + V$ representing the sum of a scalar and a vector may not be valid in the context of vector addition unless the scalar is multiplied by a basis vector or considered in a vector space where scalar addition makes sense (e.g., affine spaces).

Assumption for clarity:
Suppose that $2 + V$ is actually a vector obtained by adding the scalar 2 times some fixed vector e (say, the scalar 2 times the basis vector e), or more generally, that the notation indicates a vector $V + 2w$ where w is a fixed vector. If so, the key point is that adding a scalar to a vector directly is not valid unless we're in an affine space context.

Alternatively, if the problem intends that $2 + V$ is just another vector, then the notation is a shorthand for some vector which could be written as $V + c$ for some scalar c.
In many linear algebra contexts, the only way for {u, V, V + c} to be linearly independent is if c is zero, because $V + c$ can be expressed as $V + c\ e$, affecting linear dependence.

Conclusion:
Given typical vector space operations, the set {u, V, 2 + V} is linearly independent if and only if the vector $2 + V$ cannot be expressed as a linear combination of u and V. Since V is already in the set, the

only concern is whether *2 + V* adds a new direction or not.

Step 3: Analyzing Linear Dependence Conditions

Suppose:

$$\alpha\ u + \beta\ V + \gamma\ (2 + V) = 0$$

which simplifies to:

$$\alpha\ u + (\beta + \gamma)\ V + 2\gamma = 0$$

But as before, since 2γ is a scalar added to a vector, this indicates inconsistency unless the space allows scalar addition as a vector operation or unless the notation is a shorthand.

Assuming the last term is a vector, say, *2e*, then the equation becomes:

$$\alpha\ u + (\beta + \gamma)\ V + 2\gamma\ e = 0$$

In such a case, the set {u, V, 2 + V} is linearly independent if the only solution is $\alpha = \beta = \gamma = 0$.

Key Point:
If *2 + V* is just a vector involving a scalar multiple of some fixed vector, then since *u* and *V* are linearly independent, adding *2 + V* generally does not affect the independence unless *2 + V* can be expressed as a linear combination of *u* and *V*.

Step 4: Final Conclusion for Part (a)

- If *2 + V* is not in the span of *{u, V}*, then adding it to the set results in a linearly independent set.
- If *2 + V* can be written as a linear combination of *u* and *V*, then the set becomes linearly dependent.

Therefore:

> The set {u, V, 2 + V} is linearly independent if and only if *2 + V* is not in the span of *{u, V}*.

This final statement emphasizes that the independence depends on the specific vectors involved, especially how *2 + V* relates to *u* and *V*.

Part (b): Additional Analysis (Incomplete in the Original Prompt)

Since the original problem is incomplete (it states "Determine" but does not specify what to determine), we can explore common related questions:

Possible Interpretations:

- Determine if a specific set involving these vectors is linearly independent.
- Determine the span of the vectors.
- Find the dimension of the subspace generated by these vectors.
- Find coefficients for expressing one vector as a linear combination of others.

General Approach:
- To analyze these, one would set up equations based on the vectors and solve for scalars.
- The key steps include expressing vectors explicitly, setting up linear combinations, and solving the resulting equations.

Summary and

Frequently Asked Questions

Given that {u, V} are linearly independent, is the set {u, V, 2 + V} necessarily linearly independent?

No, the set {u, V, 2 + V} is not necessarily linearly independent. Since 2 + V can be expressed as a linear combination of V (specifically, 2 + V = V + 2), the set may be linearly dependent depending on the context.

How can we determine if the set {u, V, 2 + V} is linearly independent?

To determine if {u, V, 2 + V} is linearly independent, check if a linear combination a·u + b·V + c·(2 + V) = 0 implies that all coefficients a, b, c are zero. Simplify the combination to see if non-trivial solutions exist.

What is the significance of the set {u, V} being linearly independent in this context?

The linear independence of {u, V} implies that neither u nor V can be written as a scalar multiple of the other, providing a basis for their span. However, adding vectors like 2 + V may affect the independence depending on their relationships.

If {u, V} are linearly independent, can adding a scalar multiple of V create dependence?

Yes. Since 2 + V can be written as V + 2, which is a linear combination of V and a scalar (2), adding it to the set can introduce dependence unless 2 + V is linearly independent from u and V, which is generally not the case.

What is the process to determine the linear dependence or independence of {u, V, 2 + V}?

Express a linear combination a·u + b·V + c·(2 + V) = 0, then simplify to see if solutions other than the trivial (a=b=c=0) exist. If non-trivial solutions exist, the set is linearly dependent; otherwise, it is independent.

Can the set {u, V, 2 + V} be linearly independent if V is a zero

vector?

No. If V is the zero vector, then 2 + V = 2, which is a scalar multiple of the scalar 1, and the set would be linearly dependent since it contains the zero vector, and 2 can be expressed as a scalar multiple of V (which is zero), leading to dependence.

What are the implications of adding a vector like 2 + V to a linearly independent set {u, V}?

Adding 2 + V to {u, V} can introduce linear dependence because 2 + V can be expressed as a combination of V and a scalar (2). Therefore, the resulting set may no longer be linearly independent unless specific conditions are met.

Additional Resources

Understanding Linear Independence and Its Extensions: An In-Depth Analysis of {u, V, 2 + V}

In the realm of linear algebra, the concept of linear independence forms the cornerstone for understanding the structure of vector spaces. When analyzing sets of vectors, establishing whether they are linearly independent or dependent provides vital insights into their span, basis properties, and dimensionality. This article delves into a specific problem involving vectors {u, V} known to be linearly independent, and examines whether an augmented set, specifically {u, V, 2 + V}, maintains this independence. We will explore the problem in meticulous detail, providing comprehensive explanations, theoretical foundations, and analytical reasoning to ensure clarity and depth.

Foundations of Linear Independence

What Is Linear Independence?

Linear independence is a fundamental concept in vector spaces that describes a set of vectors that do not exhibit redundancy. Formally, a set of vectors $\{v_1, v_2, \dots, v_n\}$ in a vector space V is linearly independent if the only solution to the linear combination

$$ a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0 $$

is

$$ a_1 = a_2 = \dots = a_n = 0. $$

Conversely, if there exists a non-trivial combination (not all coefficients zero) that results in the zero vector, the set is linearly dependent.

This property is crucial because it essentially measures whether vectors in a set add new 'directions' to the span or are redundant due to being scalar multiples or linear combinations of each other.

Implications of Linear Independence

- Basis Formation: A set of linearly independent vectors that span a vector space forms a basis, dictating the dimension of the space.
- Minimal Generating Set: Linearly independent sets are minimal in the sense that removing any vector reduces the spanning capacity.
- Unique Coefficient Representation: Any vector in the span can be uniquely expressed as a linear combination of basis vectors.

Given Conditions and the Core Question

Initial Assumption: {u, V} Is Linearly Independent

The problem states that the set {u, V} is linearly independent. Here, we interpret V as a vector in a vector space (say, over a field like ℝ), and u as another vector. The independence implies:

$$ a u + b V = 0 \implies a = 0, b = 0. $$

This foundational assumption indicates that neither u nor V can be written as scalar multiples of each other, and they do not satisfy any non-trivial linear relation.

The Main Inquiry: Is {u, V, 2 + V} Linearly Independent?

The core of the problem involves determining whether adding the vector 2 + V to the set {u, V} preserves linear independence. This is a nuanced question because 2 + V is an affine combination involving the vector V, scaled by 1, and a scalar (2). The key is to analyze whether this new vector introduces a linear dependency.

Analytical Approach to the Problem

Understanding the Vector 2 + V

Before delving into independence, it is essential to interpret the vector 2 + V:

- Scalar Addition: The notation 2 + V suggests the vector V is being added to the scalar 2. In most vector space contexts, this indicates an affine shift, often in spaces like $\mathbb{R}^n$, where 2 can be considered as 2 times the scalar added to each component of V, or more generally, as a scalar multiple.

- Clarification in Vector Spaces: Typically, vectors are elements of a vector space, and scalar addition or scalar multiplication is well-defined. If V is a vector, then 2 + V is interpreted as V + 2e, where e is the scalar multiple of the identity (such as in affine spaces). Alternatively, in a purely vector space setting, 2 + V can be viewed as V + $2u_0$, where u_0 is a basis vector representing the scalar addition.

- Simplification: Assuming the context is $\mathbb{R}^n$, the notation 2 + V likely means V + 2 times some fixed scalar vector. For simplicity, and aligning with common conventions, we interpret 2 + V as V + 2 e, where e could be the vector of all ones or a particular basis vector.

In the absence of additional context, the most

straightforward interpretation is:

$$ 2 + V \equiv V + 2 \cdot \mathbf{v}_0, $$

where $\mathbf{v}_0$ is a fixed vector, possibly the zero vector or a basis vector.

For clarity, and to proceed with the analysis, we'll assume the context is such that 2 + V is an affine shift of V; that is,

$$ 2 + V = V + c, $$

where c is a scalar multiple of some fixed vector (or simply a scalar added component-wise).

Step 1: Expressing the Set {u, V, 2 + V}

The set in question is:

$$ \{ u, V, 2 + V \}. $$

Note that V and 2 + V are related. Specifically:

$$ 2 + V = V + 2 \cdot e, $$

if e is the vector of all ones, or simply, in scalar terms, this is V shifted by a scalar multiple.

This relationship suggests that V and 2 + V are linearly dependent unless the scalar shift introduces a new independent element.

Step 2: Checking for Linear Dependence

To determine if the set {u, V, 2 + V} is linearly independent, consider whether a non-trivial linear combination exists:

$$ \alpha u + \beta V + \gamma (2 + V) = 0. $$

Expanding the third term:

$$ \alpha u + \beta V + \gamma (2 + V) = \alpha u + \beta V + 2 \gamma + \gamma V. $$

Grouping like terms:

$$ (\alpha u) + (\beta V + \gamma V) + 2 \gamma = 0. $$

This simplifies to:

$$ \alpha u + (\beta + \gamma) V + 2 \gamma = 0. $$

Step 3: Analyzing the Conditions for Dependence

The key question is: can non-zero scalars α, β, γ satisfy the above equation?

- Case 1: $\alpha \neq 0$

If $\alpha \neq 0$, then:

$$ \alpha u = - (\beta + \gamma) V - 2 \gamma. $$

Given that u is linearly independent of V (by assumption), u cannot be expressed as a linear combination of V alone unless the right side involves u itself. But since the right side involves V and a scalar, and u is independent of V, the only solution here is:

$$ \alpha = 0. $$

- Case 2: $\alpha = 0$

The equation reduces to:

$$ (\beta + \gamma) V + 2 \gamma = 0. $$

Now, we analyze whether there exist scalars β, γ satisfying this.

Step 4: Solving the Reduced Equation

The equation:

$$ (\beta + \gamma) V + 2 \gamma = 0 $$

can be viewed as:

$$ (\beta + \gamma) V = - 2 \gamma. $$

Since V is a vector, and the right side is a scalar, this equality only makes sense component-wise. For the equality to hold in the vector space, the right side must be a scalar multiple of V, i.e.,

$$ - 2 \gamma = \lambda V. $$

But in general, a scalar times a vector is a vector; a scalar alone cannot equal a vector unless the scalar is zero or the vector is scalar multiple of the scalar (which usually isn't the case unless V is a scalar).

Alternatively, the equation suggests that:

- If V is the zero vector, then:

$$ (\beta + \gamma) \cdot 0 + 2 \gamma = 0 \implies 2 \gamma = 0 \implies \gamma = 0. $$

- If V is non-zero, then:

$$ (\beta + \gamma) V = - 2 \gamma. $$

Since the left side is a vector scaled by $(\beta + \gamma)$, and the right side is a scalar, the only way for the two sides to be equal is if:

$$ (\beta + \gamma) V = \text{a scalar} \times V, $$

and

$$ \text{scalar} \times V = - 2 \gamma. $$

But unless V is an eigenvector aligned with the scalar, this

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