(a) Find The Critical Numbers Of (if Any), (b)find The Open Interval(s) On Which The Function Is Increasing

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Understanding the behavior of functions is fundamental in calculus, especially when analyzing how functions change over their domains. Two essential concepts in this context are critical numbers and intervals of increase. Critical numbers help identify potential points where the function's behavior might change, such as local maxima, minima, or points of inflection. Intervals of increase, on the other hand, show where the function is rising as you move along the x-axis, providing insight into the function's growth patterns.

In this comprehensive guide, we will explore how to find the critical numbers of a function and determine the open intervals where the function is increasing. We will discuss the theoretical foundations, step-by-step procedures, common pitfalls, and practical examples to solidify your understanding.

Understanding Critical Numbers

What Are Critical Numbers?

Critical numbers (or critical points) of a function are the values of x in the domain where the derivative of the function is either zero or undefined. These points are significant because they are potential locations of local maxima, minima, or saddle points (points of inflection).

Formally, suppose f is a function differentiable on an open interval I. The critical numbers of f are all $c \in I$ such that:
- $f'(c) = 0$, or
- $f'(c)$ does not exist.

Critical numbers serve as the starting point for analyzing the function's increasing/decreasing behavior and for locating extrema.

Why Are Critical Numbers Important?

- They help identify potential local extrema.
- They are crucial in applying the First and Second Derivative Tests.
- They assist in sketching the graph of a function by highlighting key points where the function's slope

changes.

How to Find Critical Numbers

The process involves the following steps:

1. Find the derivative $f'(x)$ of the function.
2. Solve the equation $f'(x) = 0$ to find where the slope is zero.
3. Find where $f'(x)$ is undefined, provided these points are in the domain.
4. Collect all such x-values as critical numbers.

Step-by-Step Procedure for Finding Critical Numbers

Step 1: Differentiate the Function

Begin by computing the derivative $f'(x)$. Depending on the function's complexity, this may involve applying rules such as the power rule, product rule, quotient rule, or chain rule.

Example: For $f(x) = x^3 - 3x^2 + 2$, the derivative is $f'(x) = 3x^2 - 6x$.

Step 2: Solve $f'(x) = 0$

Set the derivative equal to zero and solve for x:

$$3x^2 - 6x = 0$$
$$3x(x - 2) = 0$$
$$x = 0 \quad \text{or} \quad x = 2$$

These are potential critical numbers.

Step 3: Find Points Where $f'(x)$ Is Undefined

Identify points where the derivative does not exist, such as points involving division by zero or discontinuities.

Example: For $f(x) = \frac{1}{x - 1}$, the derivative exists everywhere except at $x = 1$, where the function is undefined. So, $x=1$ is a critical number.

Step 4: Verify Critical Numbers Are in the Domain

Ensure that the critical numbers found are within the domain of the original function. If a critical number falls outside the domain, it is not valid for analysis.

Finding Intervals Where the Function Is Increasing

Understanding Intervals of Increase

A function f is said to be increasing on an interval (a, b) if for any two points $x_1, x_2 \in (a, b)$ with $x_1 < x_2$, we have:

$$f(x_1) < f(x_2)$$

Graphically, the function rises as we move from left to right within this interval.

Role of the Derivative in Determining Intervals of Increase

The First Derivative Test states:

- If $f'(x) > 0$ for every x in an interval, then f is increasing on that interval.
- If $f'(x) < 0$ for every x in an interval, then f is decreasing there.

Thus, analyzing the sign of the derivative helps identify where the function increases or decreases.

Procedure to Find Intervals of Increase

1. Identify Critical Numbers: As above, find all c where $f'(c) = 0$ or $f'(c)$ is undefined.
2. Partition the Domain: Using the critical numbers as partition points, divide the domain into intervals.
3. Test the Sign of $f'(x)$: Pick test points within each interval and evaluate $f'(x)$.
4. Determine the Behavior:
- If $f'(x) > 0$ on an interval, f is increasing there.
- If $f'(x) < 0$, f is decreasing.

Applying the Concepts Through Examples

Example 1: Finding Critical Numbers and Intervals of Increase for a Polynomial Function

Given: $f(x) = x^3 - 6x^2 + 9x + 2$

Step 1: Derive $f'(x)$:

$$f'(x) = 3x^2 - 12x + 9$$

Step 2: Set $f'(x) = 0$:

$$3x^2 - 12x + 9 = 0$$
$$x^2 - 4x + 3 = 0$$
$$(x - 1)(x - 3) = 0$$
$$x = 1 \quad \text{or} \quad x = 3$$

Step 3: Critical numbers are $x=1$ and $x=3$.

Step 4: Partition the domain into intervals:

- $(-\infty, 1)$
- $(1, 3)$
- $(3, \infty)$

Step 5: Test $f'(x)$ in each interval:

- For $x=0$ in $(-\infty, 1)$:

$$f'(0) = 3(0)^2 - 12(0) + 9 = 9 > 0$$

- For $x=2$ in $(1, 3)$:

$$f'(2) = 3(4) - 12(2) + 9 = 12 - 24 + 9 = -3 < 0$$

- For $x=4$ in $(3, \infty)$:

$$f'(4) = 3(16) - 12(4) + 9 = 48 - 48 + 9 = 9 > 0$$

Conclusion:

- f is increasing on $(-\infty, 1)$ and $(3, \infty)$.
- f is decreasing on $(1, 3)$.

Example 2: Analyzing a Rational Function

Given: $g(x) = \frac{x^2 - 4}{x - 1}$

Step 1: Find derivative $g'(x)$:

Using the quotient rule:

$$g'(x) = \frac{(2x)(x-1) - (x^2 - 4)(1)}{(x-1)^2}$$

Simplify numerator:

$$2x(x-1) - (x^2 - 4) = 2x^2 - 2x - x^2 + 4 = x^2 - 2x + 4$$

Thus,

$$g'(x) = \frac{x^2 - 2x + 4}{(x - 1)^2}$$

Step 2: Find critical points:

- Numerator $x^2 - 2x + 4$:

Discriminant: $(-2)^2 - 4 \times 1 \times 4 = 4 - 16 = -12 < 0$

Since the numerator is always positive (quadratic with positive leading coefficient and negative discriminant), $g'(x)$ is never zero.

- The derivative is undefined at $x=1$. Since $x=1$ is not in the domain (denominator zero), it is a vertical asym

Frequently Asked Questions

What is a critical number of a function?

A critical number of a function is a point in the domain where the derivative is either zero or undefined.

How do I find the critical numbers of a function?

To find critical numbers, first compute the derivative of the function, then solve for points where the derivative equals zero or does not exist within the domain.

Why are critical numbers important in analyzing a function?

Critical numbers help identify potential local maxima, minima, or points of inflection, indicating where the function's increasing or decreasing behavior may change.

What does it mean when a function is increasing on an interval?

A function is increasing on an interval if, for any two points in that interval, the function value at the second point is greater than at the first, i.e., the function's graph slopes upward.

How do I determine the intervals where a function is increasing?

Find the derivative, determine where it is positive, and then identify the corresponding open intervals on the domain where the derivative > 0.

Can a critical number be outside the interval where the function is increasing?

Yes, critical numbers can be outside the intervals of increase or decrease; they are simply points where the derivative is zero or undefined, which may or may not correspond to increasing intervals.

What role does the first derivative test play in finding increasing intervals?

The first derivative test involves analyzing the sign of the derivative around critical points to determine where the function is increasing or decreasing.

What are some common pitfalls when finding critical numbers and increasing intervals?

Common pitfalls include forgetting to check points where the derivative is undefined, miscalculating derivatives, or misinterpreting the sign of the derivative on intervals.

Can a function have no critical numbers but still be increasing?

Yes, if the derivative is positive everywhere in its domain and never zero or undefined, then the function is increasing everywhere without critical points.

How do I verify the intervals of increase after finding critical numbers?

Test points in each interval determined by the critical numbers, evaluate the derivative at those points, and check whether the derivative is positive (increasing) or negative (decreasing).

Additional Resources

Find The Critical Numbers Of (if Any), Find The Open Interval(s) On Which The Function Is Increasing

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Introduction

In the realm of calculus, understanding the behavior of functions is fundamental to analyzing and interpreting their properties. Two of the most pivotal concepts in this regard are critical numbers and the intervals of increase. Critical numbers serve as potential points where a function's behavior may change—such as local maxima, minima, or points of inflection—while intervals of increase reveal where the function ascends as we move along its domain.

This investigative article aims to thoroughly explore the process of identifying critical numbers and determining where a function is increasing. We will extend the discussion with an example, detailed steps, and insights into why these concepts are essential in calculus and applied mathematics.

Understanding Critical Numbers

Definition and Significance

A critical number of a function $f(x)$ is any point $x = c$ in the domain of f where the derivative either:

- is zero: $f'(c) = 0$, or
- does not exist: $f'(c)$ is undefined.

Critical numbers are crucial because they mark potential locations of local extrema (maxima or minima) or points of inflection—points where the function's increasing or decreasing behavior may change.

Methodology for Finding Critical Numbers

The process encompasses the following steps:

1. Determine the domain of the function: Understand where the function is defined.
2. Compute the derivative $f'(x)$: Derive the function to analyze the rate of change.
3. Solve $f'(x) = 0$: Find points where the derivative equals zero.
4. Identify where $f'(x)$ does not exist: Find points of discontinuity or non-differentiability within the domain.
5. Consolidate solutions: The union of solutions from steps 3 and 4 forms the set of critical numbers.

Analyzing the Function: A Step-by-Step Approach

Let’s consider a detailed example to illustrate these concepts in practice.

Example Function:
$$f(x) = x^3 - 6x^2 + 9x + 2$$

This polynomial function is smooth and continuous everywhere, making it an ideal candidate for analysis.

Step 1: Find the Derivative $f'(x)$

Applying basic differentiation rules:

$$
f'(x) = 3x^2 - 12x + 9
$$

Step 2: Solve $f'(x) = 0$

Set the derivative equal to zero:

$$
3x^2 - 12x + 9 = 0
$$

Divide through by 3:

$$
x^2 - 4x + 3 = 0
$$

Factorization:

$$
(x - 1)(x - 3) = 0
$$

Solutions:

$$
x = 1,\quad x = 3
$$

Step 3: Check for points where $f'(x)$ does not exist

Since $f'(x)$ is a polynomial, it exists everywhere; thus, no additional critical points arise from non-differentiability.

Critical Numbers Identified:

$$
\boxed{\text{Critical numbers are } x = 1,\quad x = 3}
$$

Determining Intervals of Increase

Having identified critical points, the next step is to analyze the function's increasing and decreasing behavior.

Step 4: Test the Sign of $f'(x)$

Choose test points in the intervals determined by the critical numbers:

- Interval 1: $x < 1$ (e.g., $x=0$)
- Interval 2: $1 < x < 3$ (e.g., $x=2$)
- Interval 3: $x > 3$ (e.g., $x=4$)

Compute $f'(x)$ at these points:

- At $x=0$:

$$
f'(0) = 3(0)^2 - 12(0) + 9 = 9 > 0
$$

- At $x=2$:

$$
f'(2) = 3(4) - 12(2) + 9 = 12 - 24 + 9 = -3 < 0
$$

- At $x=4$:

$$
f'(4) = 3(16) - 12(4) + 9 = 48 - 48 + 9 = 9 > 0
$$

Step 5: Determine the nature of these intervals

- For $x < 1$, $f'(x) > 0$: the function is increasing.
- For $1 < x < 3$, $f'(x) < 0$: the function is decreasing.
- For $x > 3$, $f'(x) > 0$: the function is increasing again.

Final Summary: Behavior of the Function

- Critical Numbers: $x=1$ and $x=3$
- Intervals of Increase:

$$
(-\infty, 1) \quad \text{and} \quad (3, \infty)
$$

- Intervals of Decrease:

$$
(1, 3)
$$

Significance of These Findings

These analyses provide vital insights:

- The function increases on $(-\infty, 1)$ and $(3, \infty)$.
- It decreases on $(1, 3)$.
- Critical points at $x=1$ and $x=3$ are candidates for local extrema:

- At $x=1$, from increasing to decreasing, indicating a local maximum.
- At $x=3$, from decreasing to increasing, indicating a local minimum.

Broader Implications and Applications

Understanding critical numbers and intervals of increase isn't solely academic; it bears significant practical implications:

- In Optimization: Critical points help identify optimal solutions in business, engineering, and data science.
- In Graphing: Recognizing where functions ascend or descend informs accurate curve sketching.
- In Physical Sciences: Derivatives relate to velocity and acceleration; critical points can denote moments of change in physical systems.
- In Economics: Critical points can indicate maximum profit or minimum cost scenarios.

Conclusion

The process of finding critical numbers and determining where a function is increasing is a cornerstone of differential calculus. By methodically computing derivatives, solving for zeros, and analyzing the sign of derivatives across the domain, mathematicians and scientists can unveil the nuanced behavior of complex functions.

This investigative approach not only enriches our understanding of mathematical properties but also empowers practical decision-making across diverse fields. Mastery of these techniques facilitates the ability to interpret, predict, and optimize real-world phenomena modeled by mathematical functions.

In summary, identifying critical numbers and open intervals of increase provides a foundational toolkit for analyzing the qualitative behavior of functions, leading to deeper insights and effective applications in both theoretical and applied contexts.

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