(a) Find The Unit Tangent And Unit Normal Vectors T(t) And N(r).(b) Use Formula 9 To Find The Curvature.17.

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Understanding the behavior of curves in space involves analyzing how they change direction and bend. Two fundamental concepts in differential geometry are the unit tangent vector $T(t)$, which indicates the direction of the curve at a point, and the unit normal vector $N(t)$, which points toward the center of curvature, indicating how the curve bends. Additionally, the curvature κ quantifies how sharply a curve bends at a given point. In this comprehensive guide, we will explore how to find the unit tangent and normal vectors for a parametric curve and how to compute the curvature using Formula 9, a standard method in differential geometry.

Understanding Curves and Their Vectors

Before diving into calculations, it's essential to understand the basic definitions:

Parametric Curves

A parametric curve in three-dimensional space is typically represented as:

$$
\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle
$$

where t is a parameter, often representing time or another variable.

Unit Tangent Vector $T(t)$

The unit tangent vector provides the direction of the curve at a point t. It is obtained by differentiating the position vector $\mathbf{r}(t)$ with respect to t, then normalizing:

\[

\boxed{
T(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}
}
\]

where \(\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} \) and \(|\mathbf{r}'(t)| \) is its magnitude.

Unit Normal Vector \(N(t) \)

The unit normal vector indicates the direction in which the curve is turning. It is orthogonal to \(T(t) \) and points toward the center of curvature:

\[
N(t) = \frac{T'(t)}{|T'(t)|}
\]

where \(T'(t) \) is the derivative of \(T(t) \) with respect to \(t \).

Step-by-Step Process to Find \(T(t) \) and \(N(t) \)

Let's now go through the steps involved in calculating these vectors for a specific parametric curve.

1. Determine \(\mathbf{r}(t) \)

Identify the parametric equations representing your curve. For example:

\[
\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle
\]

2. Compute \(\mathbf{r}'(t) \)

Differentiate each component with respect to \(t \):

\[
\mathbf{r}'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle
\]

3. Find \(|\mathbf{r}'(t)| \)

Calculate the magnitude:

\[
|\mathbf{r}'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 +
\left(\frac{dz}{dt}\right)^2}
\]

4. Compute the unit tangent vector \(T(t) \)

Normalize \(\mathbf{r}'(t) \):

\[
T(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}
\]

5. Derive \(T'(t) \)

Differentiate \(T(t) \) with respect to \(t \).

6. Compute the unit normal vector \(N(t) \)

Normalize \(T'(t) \):

\[
N(t) = \frac{T'(t)}{|T'(t)|}
\]

Applying Formula 9 to Calculate Curvature \(\kappa \)

The curvature \(\kappa \) measures how quickly a curve deviates from a straight line at a point. Formula 9, a commonly used formula in differential geometry, relates the derivatives of the position vector to the curvature:

\[
\boxed{
\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}

}
\]

where:

- \(\times \) denotes the cross product,
- \(\mathbf{r}''(t) \) is the second derivative of the position vector with respect to \(t \).

Steps to compute curvature using Formula 9:

1. Find \(\mathbf{r}''(t) \)

Differentiate \(\mathbf{r}'(t) \) with respect to \(t \).

2. Calculate the cross product \(\mathbf{r}'(t) \times \mathbf{r}''(t) \)

This vector's magnitude is crucial for the curvature.

3. Find the magnitude \(|\mathbf{r}'(t) \times \mathbf{r}''(t)| \)

Compute this scalar value.

4. Calculate \(|\mathbf{r}'(t)| \) and cube it

Cube the magnitude to prepare for the denominator.

5. Compute \(\kappa(t) \)

Divide the magnitude of the cross product by the cube of \(|\mathbf{r}'(t)| \):

\[
\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}
\]

Worked Example: Calculating \(T(t) \), \(N(t) \), and Curvature for a Sample Curve

Let's consider a specific curve to illustrate the process:

\[
\mathbf{r}(t) = \langle t, t^2, t^3 \rangle
\]

Step-by-step calculation:

1. Compute \(\mathbf{r}'(t) \):

\[
\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle
\]

2. Calculate \(|\mathbf{r}'(t)| \):

\[
|\mathbf{r}'(t)| = \sqrt{1^2 + (2t)^2 + (3t^2)^2} = \sqrt{1 + 4t^2 + 9t^4}
\]

3. Find the unit tangent vector \(T(t) \):

\[
T(t) = \frac{\langle 1, 2t, 3t^2 \rangle}{\sqrt{1 + 4t^2 + 9t^4}}
\]

4. Derive $T'(t)$.

This involves differentiating each component of $T(t)$:

$$
T'(t) = \frac{d}{dt} \left(\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right)
$$

This step can be complex, involving quotient rule or vector calculus techniques.

5. Find $N(t)$.

Normalize $T'(t)$:

$$
N(t) = \frac{T'(t)}{|T'(t)|}
$$

6. Compute $\mathbf{r}''(t)$:

$$
\mathbf{r}''(t) = \langle 0, 2, 6t \rangle
$$

7. Cross product $\mathbf{r}'(t) \times \mathbf{r}''(t)$:

\[
\mathbf{r}'(t) \times \mathbf{r}''(t) =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & 2t & 3t^2 \\
0 & 2 & 6t
\end{vmatrix}
\]

Calculating the determinant:

\[
= \mathbf{i} (2t \times 6t - 3t^2 \times 2) - \mathbf{j} (1 \times 6t - 3t^2 \times 0) + \mathbf{k} (1 \times 2 - 2t \times 0)
\]

\[
= \mathbf{i} (12t^2 - 6t^2) - \mathbf{j} (6t - 0) + \mathbf{k} (2 - 0)
\]

\[
= \mathbf{i} (6t^2) - \mathbf{j} (6t) + \mathbf{k} (2)
\]

Magnitude:

\[
|\mathbf{

Frequently Asked Questions

How do you find the unit tangent vector T(t) for a given vector function r(t)?

To find the unit tangent vector T(t), first compute the derivative r'(t), then find its magnitude |r'(t)|, and finally divide r'(t) by |r'(t)|: T(t) = r'(t)/|r'(t)|.

What is the procedure to determine the unit normal vector N(t) for a curve?

The unit normal vector N(t) is found by differentiating the unit tangent vector T(t) with respect to t, then normalizing the result: N(t) = T'(t)/|T'(t)|.

How is the curvature κ(t) calculated using Formula 9 in the context of vector functions?

Using Formula 9, the curvature κ(t) is given by κ(t) = |r'(t) × r''(t)| / |r'(t)|^3, where × denotes the cross product, r'(t) and r''(t) are the first and second derivatives of r(t).

Why is it important to normalize the tangent vector when finding the unit tangent T(t)?

Normalizing T(t) ensures it has a magnitude of 1, which makes it a true unit vector that indicates direction only, essential for accurately analyzing the curve's geometry.

In the context of curvature calculation, what role does the cross product $|r'(t) \times r''(t)|$ play?

The magnitude of the cross product $|r'(t) \times r''(t)|$ measures how sharply the curve is turning at point t, directly relating to the curvature $\kappa(t)$.

Can you explain the significance of using Formula 9 to find curvature 17, and what does the number 17 represent?

Formula 9 provides a direct way to compute curvature from derivatives of r(t). The number 17 likely refers to a specific problem number or example in a textbook, indicating the particular case where this formula is applied to find the curvature at that point.

Additional Resources

(a) Find The Unit Tangent And Unit Normal Vectors T(t) And N(r).(b) Use Formula 9 To Find The Curvature.17.

Introduction

In the realm of differential geometry, understanding the behavior of curves in space is fundamental. Key to this understanding are the concepts of tangent vectors, normal vectors, and curvature—each offering insight into the shape and nature of a curve. The task of finding the unit tangent vector $T(t)$ and the unit normal vector $N(t)$ provides a window into how a curve progresses and deviates in space. Subsequently, applying a specific formula—referred to here as Formula 9—to determine the curvature κ allows us to quantify how sharply a curve bends at a given point. This article delves into the process of deriving these vectors and calculating curvature, providing a comprehensive guide suitable for students, educators, and enthusiasts interested in the geometric properties of curves.

Understanding the Foundations: Curves, Vectors, and Their Derivatives

Before engaging with the calculations, it's essential to establish a solid understanding of the foundational concepts.

Parameterized Curves

A parametric curve in three-dimensional space is often described by a vector function:

$$
\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle
$$

where t is a real parameter that varies over an interval, producing a point $\mathbf{r}(t)$ on the curve.

Derivatives and the Tangent Vector

The derivative of $\mathbf{r}(t)$ with respect to t,

$$
\mathbf{r}'(t) = \frac{d}{dt} \mathbf{r}(t),
$$

gives the velocity vector at each point, which points in the direction of the curve's immediate motion.

Part (a): Finding the Unit Tangent Vector $T(t)$ and Unit Normal Vector $N(t)$

Step 1: Compute the Derivative $\mathbf{r}'(t)$

The first step is to differentiate each component of the vector function:

\[

\mathbf{r}'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}
\right\rangle
\]

This derivative provides a vector tangent to the curve at each point, but it may not be of unit length.

Step 2: Calculate the Magnitude \(|\mathbf{r}'(t)| \)

The magnitude of the tangent vector is:

\[
|\mathbf{r}'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 +
\left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}
\]

This scalar value indicates the speed at which the curve is traversed at \(t \).

Step 3: Derive the Unit Tangent Vector \(T(t) \)

The unit tangent vector is obtained by normalizing \(\mathbf{r}'(t) \):

\[
T(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}
\]

This vector has a magnitude of 1 and points in the direction of motion along the curve.

Example: Calculating $T(t)$

Suppose we have a curve:

$$
\mathbf{r}(t) = \langle t, t^2, t^3 \rangle
$$

- Derivative:

$$
\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle
$$

- Magnitude:

$$
|\mathbf{r}'(t)| = \sqrt{1^2 + (2t)^2 + (3t^2)^2} = \sqrt{1 + 4t^2 + 9t^4}
$$

- Unit tangent vector:

$$
T(t) = \frac{\langle 1, 2t, 3t^2 \rangle}{\sqrt{1 + 4t^2 + 9t^4}}
$$

\]

This normalized vector clearly indicates the direction of the curve at each \(t \).

Step 4: Finding the Unit Normal Vector \(N(t) \)

The unit normal vector points in the direction of the curve's principal normal, indicating how the curve bends at each point.

Step 1: Compute the derivative of \(T(t) \)

The derivative of \(T(t) \) with respect to \(t \):

\[
T'(t) = \frac{d}{dt} T(t)
\]

This derivative points towards the direction in which \(T(t) \) changes, i.e., how the direction of the tangent vector varies along the curve.

Step 2: Normalize \(T'(t) \)

The magnitude of \(T'(t) \) is:

\[

|T'(t)| = \text{scalar} \quad (\text{to be computed based on } T(t))
\]

The unit normal vector is then:

\[
N(t) = \frac{T'(t)}{|T'(t)|}
\]

It is orthogonal to \(T(t) \) and indicates the principal direction of curvature.

Practical Example: Computing \(N(t) \)

Continuing with the previous example:

- Compute \(T(t) \):

\[
T(t) = \frac{\langle 1, 2t, 3t^2 \rangle}{\sqrt{1 + 4t^2 + 9t^4}}
\]

- Derive \(T'(t) \):

This involves differentiating each component with respect to \(t \) and applying the quotient rule or chain rule as necessary.

- Normalize $T'(t)$:

Calculate its magnitude and divide to obtain $N(t)$.

Part (b): Using Formula 9 to Find the Curvature κ

Once $T(t)$ and $N(t)$ are known, the curvature κ at each point on the curve can be computed. The general formula for curvature in terms of derivatives is:

$$
\kappa(t) = \frac{| \mathbf{r}'(t) \times \mathbf{r}''(t) |}{|\mathbf{r}'(t)|^3}
$$

This is often referred to as Formula 9 in many textbooks and resources, emphasizing the geometric relationship between the derivatives of the position vector.

Deriving the Curvature Step-by-Step

Step 1: Compute $\mathbf{r}''(t)$

Differentiate $\mathbf{r}'(t)$:

$$
\mathbf{r}''(t) = \frac{d^2}{dt^2} \mathbf{r}(t)
$$

For the example:

$$
\mathbf{r}''(t) = \left\langle 0, 2, 6t \right\rangle
$$

Step 2: Calculate the cross product $\mathbf{r}'(t) \times \mathbf{r}''(t)$

This vector operation captures the rate at which the tangent vector changes direction, which relates directly to curvature.

For the example:

$$
\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle
$$

$$
\mathbf{r}''(t) = \langle 0, 2, 6t \rangle
$$

The cross product:

\[

\mathbf{r}'(t) \times \mathbf{r}''(t) =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & 2t & 3t^2 \\
0 & 2 & 6t
\end{vmatrix}
\]

Calculating the determinant yields:

\[
\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle (2t)(6t) - (3t^2)(2), -[1 \cdot 6t - 3t^2 \cdot 0], 1 \cdot 2 - 2t \cdot 0 \rangle
\]

Simplify:

\[
= \langle 12t^2 - 6t^2, - (6t - 0), 2 - 0 \rangle = \langle 6t^2, -6t, 2 \rangle
\]

Step 3: Calculate the magnitude of the cross product

\[
| \mathbf{r}'(t) \times \mathbf{r}''(t) | = \sqrt{(6t^2)^2 + (-6t)^2 + 2^2} = \sqrt{36t^4 + 36t^2 + 4}
\]

Step 4: Compute the magnitude \(|\mathbf{r}'(t)| \)

As previously calculated:

$$
|\mathbf{r}'(t)| = \sqrt{1 + 4t^2 + 9t^4}
$$

Step 5: Apply Formula

A Find The Unit Tangent And Unit Normal Vectors T T And N R B Use Formula 9 To Find The Curvature 17

A Find The Unit Tangent And Unit Normal Vectors T T And N R B Use Formula 9 To Find The Curvature 17

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