A Particle In The Infinite Square Well Has The Initial Wave Function (x,0) = {Ax, 0 < X < A/2{A(a-x),

Introduction

A particle in the infinite square well has the initial wave function (x,0) = {Ax, 0 < x < A/2; A(a - x), A/2 < x < A}. This initial wave function describes a quantum particle confined within a one-dimensional potential well, where the potential is zero inside the well (from x=0 to x=A) and infinite outside this region. The shape of the initial wave function is piecewise linear, increasing from zero at x=0 to a maximum at x=A/2, then decreasing back to zero at x=A. Understanding the evolution of this wave function over time requires expanding it in terms of the well's eigenstates, calculating the coefficients for this expansion, and analyzing the time-dependent behavior.

This article explores the mathematical formulation, the process of decomposing the initial state into eigenstates, and the physical interpretation of the wave function's evolution within the well. We will also discuss the implications for probability densities and expectation values, providing a comprehensive understanding of the system.

The Infinite Square Well and Its Eigenstates

Potential and Boundary Conditions

In the infinite square well, the potential V(x) is defined as:

- V(x) = 0 for 0 < x < A
- V(x) = ∞ for x ≤ 0 or x ≥ A

These boundary conditions enforce that the wave function must vanish at the walls:

- ψ(0, t) = 0
- ψ(A, t) = 0

This leads to discrete energy eigenstates characterized by quantum numbers n = 1, 2, 3, ...

Eigenfunctions and Eigenvalues

The normalized stationary eigenfunctions for the infinite square well are well-known:

$$
\phi_n(x) = \sqrt{\frac{2}{A}} \sin \left(\frac{n \pi x}{A} \right), \quad n=1,2,3,\dots
$$

with corresponding energy eigenvalues:

$$
E_n = \frac{n^2 \pi^2 \hbar^2}{2 m A^2}
$$

where:

- $\hbar$ is the reduced Planck's constant,
- m is the mass of the particle,
- A is the width of the well.

Any initial wave function $\psi(x, 0)$ can be expanded as:

$$
\psi(x, 0) = \sum_{n=1}^\infty c_n \phi_n(x)
$$

where the coefficients c_n are determined by projecting the initial wave function onto each eigenstate.

Initial Wave Function Analysis

Definition of the Wave Function

The initial wave function is given as a piecewise linear function:

$$
\psi(x,0) =
\begin{cases}
A x, & 0 < x < \frac{A}{2} \\
A \left(a - x \right), & \frac{A}{2} < x < A \\
0, & \text{elsewhere}
\end{cases}
$$

where A is a normalization constant and a is a parameter that influences the shape. To proceed, we need to determine A and a such that $\psi(x,0)$ is normalized.

Normalization of the Wave Function

The normalization condition requires:

$$
\int_0^A |\psi(x,0)|^2 dx = 1
$$

Calculating this integral:

$$
\int_0^{A/2} (A x)^2 dx + \int_{A/2}^A [A (a - x)]^2 dx = 1
$$

which simplifies to:

$$
A^2 \left(\int_0^{A/2} x^2 dx + \int_{A/2}^A (a - x)^2 dx \right) = 1
$$

Computing the integrals:

- First integral:

$$
\int_0^{A/2} x^2 dx = \frac{1}{3} \left(\frac{A}{2} \right)^3 = \frac{A^3}{24}
$$

- Second integral:

Let’s evaluate:

$$
\int_{A/2}^A (a - x)^2 dx
$$

Substitute $u = a - x$, then $du = - dx$, and when $x = A/2$, $u = a - A/2$; when $x = A$, $u = a - A$.

The integral becomes:

$$
\int_{u = a - A}^{a - A/2} u^2 (-du) = \int_{a - A/2}^{a - A} u^2 du
$$

which yields:

$$
\frac{1}{3} u^3 \Big|_{a - A}^{a - A/2} = \frac{1}{3} \left[(a - A/2)^3 - (a - A)^3 \right]
$$

Putting it all together:

$$
A^2 \left(\frac{A^3}{24} + \frac{1}{3} \left[(a - A/2)^3 - (a - A)^3 \right] \right) = 1
$$

From here, A and a can be chosen or determined based on normalization constraints.

Calculating Expansion Coefficients c_n

Projection onto Eigenstates

The expansion coefficients are given by:

$$
c_n = \int_0^A \phi_n^(x) \psi(x,0) dx
$$

Since the eigenfunctions are real, this simplifies to:

$$
c_n = \sqrt{\frac{2}{A}} \left[\int_0^{A/2} x \sin \left(\frac{n \pi x}{A} \right) dx + \int_{A/2}^A (a - x) \sin \left(\frac{n \pi x}{A} \right) dx \right]
$$

Calculating these integrals involves integration by parts or lookup of standard integrals. The process is as follows:

- For each integral, express the integrand as the product of a polynomial and sine function.
- Use integration by parts to evaluate:
- $\int x \sin(kx) dx$,
- $\int (a - x) \sin(kx) dx$.

The resulting c_n coefficients encapsulate how the initial wave function decomposes into the eigenstates.

Analytical or Numerical Approach

Given the piecewise linear form, explicit formulas for c_n may be complex. Therefore, practical calculations often involve numerical integration:

- Discretize the domain,
- Evaluate the integrand at sufficient points,
- Use numerical methods such as Simpson’s rule or Gaussian quadrature.

This approach provides approximate c_n values, which can then be used to analyze the time evolution.

Time Evolution of the Wave Function

General Expression

The wave function at any time t can be reconstructed as:

$$
\psi(x, t) = \sum_{n=1}^\infty c_n \phi_n(x) e^{-i E_n t / \hbar}
$$

where:

- c_n are the expansion coefficients,
- $\phi_n(x)$ are the eigenfunctions,
- E_n are the eigenenergies.

This superposition encapsulates the quantum dynamics within the well.

Physical Interpretation

The wave function's time evolution leads to phenomena such as:

- Quantum interference among eigenstates,
- Wave packet spreading,
- Revival and fractional revival patterns.

The initial linear shape influences the distribution of energy eigenstates, affecting how quickly the wave function disperses and how the probability density evolves.

Probability Density and Expectation Values

Probability Density

The probability density at time t is:

$$
|\psi(x, t)|^2 = \left| \sum_{n=1}^\infty c_n \phi_n(x) e^{-i E_n t / \hbar} \right|^2
$$

This quantity reveals where the particle is most likely to be found at any given moment.

Expectation Values

Key expectation values include:

- Position:

$$
\langle x \rangle (t) = \int_0^A x |\psi(x, t)|^2 dx
$$

- Momentum:

$$
\langle p \rangle (t) = \int_0^A \psi^(x, t) \left(-i \hbar \frac{\partial}{\partial x} \right) \psi(x, t) dx
$$

These provide insights into the particle's average position and momentum over time.

Implications and Applications

Understanding the evolution of this particular

Frequently Asked Questions

What is the initial wave function for a particle in an infinite square well as given by (x,0) = {A x, 0 < x < A/2; A (A - x), A/2 < x < A}?

The initial wave function describes a piecewise function where it increases linearly from zero at x=0 to a maximum at x=A/2, then decreases linearly back to zero at x=A, forming a symmetric 'tent'-shaped profile within the well.

How can the initial wave function be normalized in this infinite square well problem?

Normalization involves integrating the square of the wave function over the well's domain and setting the integral equal to one. For the given piecewise function, you compute the integrals of $(A x)^2$ and $(A (A - x))^2$ over their respective intervals and sum them, then solve for the normalization constant A.

What are the key steps to find the time evolution of this initial

wave function in the infinite square well?

First, expand the initial wave function into the infinite well's eigenstates by calculating the expansion coefficients. Then, multiply each eigenstate by a phase factor involving its energy and time to find the wave function at later times. Summing these contributions yields the time-evolved wave function.

How does the initial wave function's shape influence its energy eigenstate expansion in the well?

The tent-shaped initial wave function can be expressed as a linear combination of the infinite square well's eigenstates. Its shape determines the distribution of expansion coefficients, with more complex shapes generally requiring higher-energy eigenstates for an accurate representation.

What physical insights can be gained from analyzing the initial wave function (x,0) = {A x, 0 < x < A/2; A (A - x), A/2 < x < A}?

This wave function represents a particle with a non-uniform probability distribution, peaked at the center of the well. Studying its evolution reveals how localized or delocalized states evolve over time, illustrating quantum superposition, interference, and the spreading of wave packets.

What challenges arise when calculating the expectation values of position and momentum for this initial wave function?

Calculating expectation values involves integrating the wave function multiplied by the relevant operators. The piecewise nature of the wave function can lead to complex integrals, especially for momentum, which requires derivatives. Ensuring proper normalization and handling the discontinuities at x=A/2 are also important considerations.

Additional Resources

A Particle In The Infinite Square Well Has The Initial Wave Function (x,0) = {Ax, 0 < x < A/2; A(a - x),

Introduction

The quantum mechanical behavior of particles confined within potential wells has been a cornerstone of quantum physics research. Among these, the infinite square well (also known as the particle in a box) is perhaps the most fundamental model, providing invaluable insights into quantum states, energy quantization, and wave function evolution. The initial wave function of a particle in this well significantly influences its subsequent dynamics, observable properties, and the way it interacts with measurements. In this review, we examine a particular initial wave function defined as:

\[
\psi(x, 0) =
\begin{cases}
A x, & 0 < x < \frac{A}{2} \\

A (a - x), & \frac{A}{2} < x < A \\
0, & \text{elsewhere}
\end{cases}
\]

This piecewise linear wave function, characterized by a peak at the center of the well and symmetric decay toward the edges, presents an interesting case for analysis within the infinite square well framework. We will explore its mathematical form, normalization, expansion into stationary states, time evolution, and physical implications.

Mathematical Description and Normalization

The Initial Wave Function

The specified initial wave function is piecewise linear, symmetric about the midpoint $x = A/2$, peaking at $x = A/2$. Its form is:

$$
\psi(x, 0) =
\begin{cases}
A x, & 0 < x < \frac{A}{2} \\
A (A - x), & \frac{A}{2} < x < A \\
0, & \text{elsewhere}
\end{cases}
$$

This function resembles a "tent" shape, with the maximum at $x = A/2$:

$$
\psi\left(\frac{A}{2}, 0\right) = A \times \frac{A}{2} = \frac{A^2}{2}
$$

It is continuous across $x = A/2$, symmetric, and vanishes at the boundaries $x=0$ and $x=A$. The symmetry simplifies analysis, especially when expanding into the eigenstates of the infinite square well.

Normalization of the Wave Function

To ensure the wave function is physically acceptable, it must be normalized:

$$
\int_0^A |\psi(x, 0)|^2 dx = 1
$$

Calculating the normalization constant A:

$$
\int_0^{A/2} (A x)^2 dx + \int_{A/2}^A (A (A - x))^2 dx = 1
$$

Computing the integrals:

\[
A^2 \int_0^{A/2} x^2 dx + A^2 \int_{A/2}^A (A - x)^2 dx = 1
\]

1. First integral:

\[
A^2 \times \frac{(A/2)^3}{3} = A^2 \times \frac{A^3/8}{3} = A^2 \times \frac{A^3}{24} = \frac{A^5}{24}
\]

2. Second integral:

Change variable \(y = A - x \), when \(x = A/2 \), \(y = A/2 \); when \(x = A \), \(y = 0 \):

\[
A^2 \int_{A/2}^A (A - x)^2 dx = A^2 \int_0^{A/2} y^2 dy = A^2 \times \frac{(A/2)^3}{3} = \frac{A^5}{24}
\]

Adding both:

\[
\frac{A^5}{24} + \frac{A^5}{24} = \frac{A^5}{12}
\]

Set equal to 1:

\[
\frac{A^5}{12} = 1 \Rightarrow A^5 = 12 \Rightarrow A = (12)^{1/5}
\]

Thus, the normalization constant A is:

\[
A = \left(12\right)^{1/5}
\]

This normalization ensures the wave function's total probability is unity.

Expansion in Terms of Stationary Eigenstates

Eigenstates of the Infinite Square Well

The stationary states for a particle in an infinite potential well of width \(A\) are:

\[

\phi_n(x) = \sqrt{\frac{2}{A}} \sin \left(\frac{n \pi x}{A} \right), \quad n=1, 2, 3, \ldots
\]

with corresponding energy eigenvalues:

\[
E_n = \frac{\hbar^2 \pi^2 n^2}{2 m A^2}
\]

Coefficients of Expansion

Expressing the initial wave function as a superposition:

\[
\psi(x, 0) = \sum_{n=1}^\infty c_n \phi_n(x)
\]

The coefficients are given by:

\[
c_n = \int_0^A \phi_n^(x) \psi(x,0) dx
\]

Given the symmetry of \(\psi(x, 0) \), only sine functions contribute (since \(\phi_n \) are sine functions), simplifying the calculation.

Calculating \(c_n \):

\[
c_n = \sqrt{\frac{2}{A}} \left[\int_0^{A/2} \sin \left(\frac{n \pi x}{A} \right) A x dx + \int_{A/2}^A \sin \left(\frac{n \pi x}{A} \right) A (A - x) dx \right]
\]

The explicit evaluation involves integrating products of \(x \sin(kx) \), which can be performed via integration by parts or tabulated integrals.

Numerical and Analytical Approaches

Analytical solutions for \(c_n \) involve multiple integrations and simplifications. For practical purposes, numerical methods or approximation schemes (such as truncating the expansion at a finite \(n \)) are often employed.

Time Evolution of the Wave Function

The time-dependent wave function is:

\[
\psi(x,t) = \sum_{n=1}^\infty c_n \phi_n(x) e^{-i E_n t / \hbar}
\]

This superposition leads to the evolution of the initial wave packet, exhibiting phenomena such as wave packet spreading, interference effects, and revival patterns.

Wave Packet Dynamics

Given the initial wave function's shape, the wave packet is initially localized with a peak at the center. Over time, due to differences in energy eigenvalues E_n, the components acquire different phase factors, leading to the dispersion of the wave packet.

Revival and Collapse

The quadratic dependence of E_n on n means that the wave function may experience partial or full revivals at specific times:

$$
T_{\text{rev}} = \frac{4 m A^2}{\hbar \pi}
$$

At fractional multiples of T_{rev}, fractional revivals and intricate interference patterns can occur.

Physical Implications and Observables

Expectation Values

- Position: The expectation value $\langle x \rangle$ oscillates over time, reflecting the initial localization and subsequent spreading.

- Momentum: The momentum distribution, obtained via the Fourier transform of $\psi(x,t)$, reveals the wave packet's spread and the presence of interference fringes.

Probability Density Evolution

The time-dependent probability density $|\psi(x,t)|^2$ displays characteristic quantum behavior:

- Initial localization at the center.
- Spreading and interference effects.
- Periodic revivals, depending on the well's parameters.

Measurement Considerations

The shape of the initial wave function influences measurement outcomes, such as the likelihood of detecting the particle at various positions over time. Its symmetry and peaked structure make it suitable for studying quantum coherence and decoherence phenomena.

Pros and Cons of the Chosen Initial Wave Function

Pros

- Symmetry: The symmetric form simplifies calculations and provides clear insights into the evolution.
- Physical Intuition: The "tent" shape models a localized particle with a maximum at the center, resembling a realistic wave packet.
- Analytical Tractability: Piecewise linear functions are easier to handle in integrals than more complex shapes.

Cons

- Discontinuities in Derivatives: Although the wave function is continuous, its derivatives are not, which can introduce subtleties in the evolution and numerical simulations.
- Limited Localization: The linear shape may not represent the most sharply localized or Gaussian-like wave packets common in experiments.
- Approximate Nature: Real physical systems often have more complex initial states; this model is idealized.

Features and Significance

This initial wave function exemplifies how a simple, symmetric, and piecewise linear state evolves within a confined quantum system. It demonstrates fundamental concepts like state expansion, superposition, and wave packet evolution, making it an excellent pedagogical tool. Its analytical properties serve as a foundation for more complex

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