An Arithmetic Sequence Has Common Difference D. The Series Sums S2, S5 And S7 Themselves Form An Arithmetic

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Understanding the properties of arithmetic sequences is fundamental in mathematics, especially when dealing with series and their sums. One intriguing aspect is when the sums of certain terms within an arithmetic sequence themselves form an arithmetic progression. In particular, if an arithmetic sequence has a common difference D, and the sums of the first 2, 5, and 7 terms—denoted as S2, S5, and S7—form an arithmetic series, it opens up interesting avenues for exploration and problem-solving. This article delves into this fascinating topic, explaining the concepts, deriving key formulas, and providing practical examples to illustrate how these ideas come together.

Understanding Arithmetic Sequences and Series

What Is an Arithmetic Sequence?

An arithmetic sequence is a sequence of numbers where each term after the first is obtained by adding a constant difference, D, to the previous term. Mathematically, the sequence can be expressed as:

- Term n: $a_n = a_1 + (n - 1)D$

where:

- a_1 is the first term
- D is the common difference
- n is the term number

Sum of the First n Terms

The sum of the first n terms of an arithmetic sequence, denoted as S_n, can be calculated using the formula:

- $S_n = \frac{n}{2} [2a_1 + (n - 1)D]$

This formula is fundamental when analyzing properties related to series sums.

Series Sums S2, S5, and S7 Forming an Arithmetic Progression

Defining the Series Sums

Given an arithmetic sequence with first term a_1 and common difference D, the sums are:

- $S_2 = \frac{2}{2} [2a_1 + (2 - 1)D] = a_1 + D$
- $S_5 = \frac{5}{2} [2a_1 + (5 - 1)D] = \frac{5}{2} [2a_1 + 4D]$
- $S_7 = \frac{7}{2} [2a_1 + (7 - 1)D] = \frac{7}{2} [2a_1 + 6D]$

Condition for Sums to Form an Arithmetic Progression

The key question is: under what conditions do $S_2, S_5,$ and S_7 themselves form an arithmetic progression?

Recall that a sequence T_1, T_2, T_3 is an arithmetic progression if:

- $2T_2 = T_1 + T_3$

Applying this to our sums:

- Check if $2S_5 = S_2 + S_7$

If this condition holds, then the sums $S_2, S_5,$ and S_7 form an arithmetic sequence.

Deriving the Conditions for the Sums to Form an Arithmetic Progression

Expressing $S_2, S_5,$ and S_7 in Terms of a_1 and D

Let's explicitly write out these sums:

- $S_2 = a_1 + D$
- $S_5 = \frac{5}{2} (2a_1 + 4D) = \frac{5}{2} \times 2 (a_1 + 2D) = 5(a_1 + 2D)$
- $S_7 = \frac{7}{2} (2a_1 + 6D) = \frac{7}{2} \times 2 (a_1 + 3D) = 7(a_1 + 3D)$

Applying the Arithmetic Progression Condition

The condition $2S_5 = S_2 + S_7$ becomes:

$$
2 \times 5(a_1 + 2D) = (a_1 + D) + 7(a_1 + 3D)
$$

Simplify both sides:

$$
10(a_1 + 2D) = a_1 + D + 7a_1 + 21D
$$

$$
10a_1 + 20D = (a_1 + 7a_1) + (D + 21D)
$$

$$
10a_1 + 20D = 8a_1 + 22D
$$

Bring all terms to one side:

$$
10a_1 + 20D - 8a_1 - 22D = 0
$$

$$
(10a_1 - 8a_1) + (20D - 22D) = 0
$$

$$
2a_1 - 2D = 0
$$

Divide through by 2:

$$
a_1 - D = 0
$$

Therefore:

$$
a_1 = D
$$

Key Result: For the sums $S_2, S_5,$ and S_7 to form an arithmetic sequence, the first term must equal the common difference, i.e., $a_1 = D$.

Implications of the Condition $a_1 = D$

Expressing the Sequence and Sums When $a_1 = D$

If $a_1 = D$, then:

- The terms of the sequence are:

- $a_n = a_1 + (n - 1)D = D + (n - 1)D = Dn$

- The sums become:

- $S_2 = a_1 + D = D + D = 2D$
- $S_5 = 5(a_1 + 2D) = 5(D + 2D) = 5(3D) = 15D$
- $S_7 = 7(a_1 + 3D) = 7(D + 3D) = 7(4D) = 28D$

Check if these sums form an arithmetic progression:
$$
2 \times 15D = 30D
$$
and
$$
S_2 + S_7 = 2D + 28D = 30D
$$

which satisfies the condition $2S_5 = S_2 + S_7$.

Thus, when $a_1 = D$, the sums $S_2, S_5,$ and S_7 indeed form an arithmetic progression.

Practical Examples and Applications

Example 1: Find the sequence and verify the sums

Suppose $a_1 = D = 3$. The sequence is:

- $a_n = 3n$

The sums are:

- $S_2 = 2 \times 3 = 6$
- $S_5 = 15 \times 3 = 45$
- $S_7 = 28 \times 3 = 84$

Check if these sums form an arithmetic sequence:
$$
2 \times 45 = 90
$$
and
$$
6 + 84 = 90
$$
which confirms that the sums form an arithmetic progression.

Application in Problem Solving

This property is useful in various mathematical problems, especially in:

- Sequence Analysis: Determining the relationship between terms and sums.
- Series Summation: Simplifying complex series by leveraging the conditions when sums themselves form an arithmetic sequence.
- Mathematical Proofs: Demonstrating properties of sequences and series based on initial conditions.

Conclusion

In summary, when dealing with an arithmetic sequence with common difference D, the sums of the first 2, 5, and 7 terms—$S_2, S_5,$ and S_7—will themselves form an arithmetic progression if and only if the first term a_1 equals the common difference D. Under this condition, the sequence simplifies to $a_n = Dn$, and the

Frequently Asked Questions

What is an arithmetic sequence and how is it characterized?

An arithmetic sequence is a sequence of numbers in which each term differs from the previous one by a constant value called the common difference D.

How are the series sums S2, S5, and S7 related in an arithmetic sequence?

In an arithmetic sequence, the sums of the first 2, 5, and 7 terms (S2, S5, S7) are themselves in arithmetic progression, meaning the difference between consecutive sums is constant.

What is the formula for the sum of the first n terms of an arithmetic sequence?

The sum S_n of the first n terms is given by $S_n = n/2 [2a + (n-1)D]$, where a is the first term and D is the common difference.

If S2, S5, and S7 form an arithmetic sequence, what condition must the sums satisfy?

They must satisfy the condition that twice the middle sum equals the sum of the first and third sums: 2S5 = S2 + S7.

How can the common difference D be determined from the sums S2, S5, and S7?

By expressing S2, S5, and S7 in terms of a and D, and using the arithmetic progression condition, D can be solved from the equations involving these sums.

Is it always true that S2, S5, and S7 form an arithmetic sequence in any arithmetic progression?

No, it depends on the specific values of the first term and common difference; the sums will form an arithmetic sequence only under certain conditions related to these parameters.

Can the concept of series sums forming an arithmetic sequence be applied to geometric series?

No, this property is specific to arithmetic sequences; geometric series have different sum behaviors and do not generally produce sums forming an arithmetic progression.

What practical applications can understanding the sums S2, S5, and S7 in an arithmetic sequence have?

This understanding can be useful in financial calculations, pattern recognition, and solving problems involving linear progressions in various fields like engineering and economics.

Additional Resources

An Arithmetic Sequence Has Common Difference D. The Series Sums S_2, S_5, and S_7 Themselves Form an Arithmetic Sequence is a fascinating topic in the realm of algebra and sequence analysis. It combines the fundamental concepts of arithmetic progressions with the intriguing properties of their partial sums. This exploration not only deepens understanding of sequences but also demonstrates the elegance and interconnectedness of mathematical structures. In this comprehensive review, we will analyze the key ideas, mathematical derivations, and implications of this statement, providing clarity and insight for students, educators, and enthusiasts alike.

Understanding Arithmetic Sequences and Series

Before delving into the specific problem, it is essential to establish a clear understanding of arithmetic sequences and their series.

Arithmetic Sequence Basics

An arithmetic sequence is a sequence of numbers in which each term after the first is obtained by adding a constant difference, D, to the previous term. Formally, if the first term is a_1, then the n-th term is:

$$a_n = a_1 + (n - 1)D$$

Features:
- Constant difference D.
- Terms progress linearly.
- Easy to generate terms and analyze.

Pros:
- Simplicity in understanding and calculating terms.
- Closed-form expressions for terms and sums.

Cons:
- Limited variability; only linear progression.
- Not suitable for modeling non-linear phenomena without modifications.

Series and Partial Sums

The series corresponding to an arithmetic sequence sums its terms:

$$S_n = a_1 + a_2 + \ldots + a_n$$

The sum of the first n terms of an arithmetic sequence is given by:

$$S_n = \frac{n}{2} [2a_1 + (n - 1)D]$$

This formula highlights a linear relation in n and D, making the analysis of sums straightforward.

The Main Problem: The Series Sums S_2, S_5, and S_7 Form an Arithmetic Sequence

The core statement asserts that:

> The sums of the first 2, 5, and 7 terms of an arithmetic sequence, denoted $S_2, S_5, \text{ and } S_7$, themselves form an arithmetic sequence.

Mathematically, this means:

$$S_5 - S_2 = S_7 - S_5$$

or equivalently,

$$2S_5 = S_2 + S_7$$

This condition introduces a secondary layer of structure: the sums themselves exhibit an arithmetic progression, revealing deeper properties of the original sequence.

Deriving the Conditions for Sums to Form an Arithmetic Sequence

To analyze this, we need explicit expressions for $S_2, S_5, \text{ and } S_7$.

Expressing the Partial Sums

Given the general sum formula:

$$S_n = \frac{n}{2} [2a_1 + (n - 1)D]$$

we have:

- $S_2 = \frac{2}{2} [2a_1 + (2 - 1)D] = 1 \times [2a_1 + D] = 2a_1 + D$
- $S_5 = \frac{5}{2} [2a_1 + (5 - 1)D] = \frac{5}{2} [2a_1 + 4D]$
- $S_7 = \frac{7}{2} [2a_1 + (7 - 1)D] = \frac{7}{2} [2a_1 + 6D]$

Calculating explicitly:

$$ S_5 = \frac{5}{2} (2a_1 + 4D) = \frac{5}{2} \times 2a_1 + \frac{5}{2} \times 4D = 5a_1 + 10D $$

$$ S_7 = \frac{7}{2} (2a_1 + 6D) = 7a_1 + 21D $$

Condition for Sums to Form an Arithmetic Sequence

Using the property:

$$ 2S_5 = S_2 + S_7 $$

Substitute the expressions:

$$ 2(5a_1 + 10D) = (2a_1 + D) + (7a_1 + 21D) $$

Simplify both sides:

Left side:

$$ 10a_1 + 20D $$

Right side:

$$ 2a_1 + D + 7a_1 + 21D = (2a_1 + 7a_1) + (D + 21D) = 9a_1 + 22D $$

Set equal:

$$ 10a_1 + 20D = 9a_1 + 22D $$

Rearranged:

$$ 10a_1 - 9a_1 = 22D - 20D $$

$$ a_1 = 2D $$

Result:

The initial term a_1 must be twice the common difference D for the sums $S_2, S_5, \text{ and } S_7$ to form an arithmetic sequence.

Implications and Further Analysis

Having established $a_1 = 2D$, it's insightful to analyze the properties of the original sequence and the sums.

Expressing the Sequence with the Derived Condition

The sequence terms:

$$a_n = a_1 + (n - 1)D = 2D + (n - 1)D = D(n + 1)$$

So, the sequence simplifies to:

$$a_n = D(n + 1)$$

This is a sequence where each term increases linearly with n, scaled by D, starting from $a_1 = 2D$.

Calculating the Sums Explicitly

Using the simplified form:

- $S_2 = 2a_1 + D = 2 \times 2D + D = 4D + D = 5D$
- $S_5 = 5a_1 + 10D = 5 \times 2D + 10D = 10D + 10D = 20D$
- $S_7 = 7a_1 + 21D = 7 \times 2D + 21D = 14D + 21D = 35D$

Check the arithmetic progression:

$$S_5 - S_2 = 20D - 5D = 15D$$

$$S_7 - S_5 = 35D - 20D = 15D$$

Confirmed: the sums are equally spaced, with a common difference of $15D$.

Features:

- The sums are proportional to D.

- The sums form an arithmetic sequence with a common difference of $15D$.

Broader Implications and Applications

This analysis reveals fundamental properties of arithmetic sequences and their partial sums, with applications across various fields.

Educational Significance

- Demonstrates how conditions on sums can constrain the initial term and difference.
- Reinforces understanding of series, sums, and sequence behavior.
- Provides a concrete example of reverse-engineering sequence parameters from sum properties.

Mathematical and Applied Contexts

- Useful in problems requiring the design of sequences with specific sum properties.
- Applicable in financial mathematics where installment plans or linear growth models are analyzed.
- Can model phenomena with predictable incremental changes, such as population growth or resource accumulation.

Pros and Cons of This Approach

Pros:

- Clarity and Simplicity: The derivation is straightforward, relying on fundamental formulas.
- Generalizability: The approach can be extended to other sums or sequences.
- Educational Value: Highlights the power of algebraic manipulation and understanding sequence properties.

Cons:

- Limited to Arithmetic Sequences: The analysis hinges on the linear nature of arithmetic progressions.

- Assumption of Non-zero D: The derivation assumes D is not zero; special cases require separate treatment.
- Abstract Nature: May seem detached from real-world data unless contextualized properly.

Conclusion

The exploration of An Arithmetic Sequence Has Common Difference D. The Series Sums S_2, S_5, and S_7 Themselves Form an Arithmetic sequence reveals a beautiful interplay between sequence parameters and their partial sums. The key takeaway is that for the sums \(S_2, S_5, \text{ and } S_7 \) to be in arithmetic progression, the initial term must satisfy \(a_1 = 2D \). This condition simplifies the sequence to \(a_n = D(n + 1) \), a sequence with a clear linear pattern and predictable sum behavior.

This analysis not only enhances understanding of the structural properties of arithmetic sequences but also provides practical tools for designing sequences with specific sum characteristics. It exemplifies how algebraic manipulation and logical reasoning can uncover

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