(b) (5 Marks) Let V Be A Vector Space, And Let S Be A Nonempty Subset Of V. Prove The Following Statement:

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Introduction to Vector Spaces and Subsets

In the realm of linear algebra, understanding the structure and properties of vector spaces is fundamental. A vector space V over a field F is a set equipped with two operations—vector addition and scalar multiplication—that satisfy certain axioms such as associativity, commutativity, existence of additive identity and inverses, distributivity, and compatibility with scalar multiplication. These properties facilitate the analysis and manipulation of vectors in diverse mathematical and applied contexts.

Within a vector space V, subsets play a critical role in understanding substructures that inherit the properties of V. A particularly important subset is a subspace, which itself is a vector space under the same operations as V. To classify a subset $S \subseteq V$ as a subspace, it must satisfy specific criteria, which are often verified through proof.

This article aims to rigorously prove a fundamental property related to subsets of vector spaces, particularly focusing on the conditions under which a subset S of V forms a subspace. The statement in question, often encountered in linear algebra courses, is:

> "If S is a nonempty subset of V such that for all $u, v \in S$ and all scalars $c \in F$, the vector $u + c v$ also belongs to S, then S is a subspace of V."

Our goal is to analyze this statement, understand its implications, and provide a formal proof, establishing the necessary and sufficient conditions for S to be a subspace based on the given properties.

Understanding the Statement and Its Significance

Before delving into the proof, it is essential to interpret the statement correctly. It asserts that:

- V is a vector space over a field F.
- S is a nonempty subset of V.
- For every $u, v \in S$ and every scalar $c \in F$, the vector $u + c v$ is in S.

This property is quite powerful because it encompasses both the closure under vector addition and scalar multiplication, but in a specific combined form involving linear combinations of the vectors u and v.

Why is this important?
In linear algebra, the characterization of subspaces often involves closure properties:

- Closure under vector addition: For all $u, v \in S$, $u + v \in S$.
- Closure under scalar multiplication: For all $u \in S$ and $c \in F$, $c u \in S$.

The statement in question effectively combines these properties into a single condition, enabling a more straightforward proof that S is a subspace.

Preliminaries and Definitions

To proceed, we recall some foundational definitions:

Definition of a Subspace

A subset $S \subseteq V$ is called a subspace of V if:

- S is nonempty.
- S is closed under vector addition: for all $u, v \in S$, $u + v \in S$.
- S is closed under scalar multiplication: for all $u \in S$ and $c \in F$, $c u \in S$.

Once these properties are established, S inherits the vector space structure from V.

Equivalent Characterizations of Subspaces

It is often easier to verify subspace properties by using alternative criteria, such as:

- If S is nonempty and closed under linear combinations of its vectors, then S is a subspace.
- Specifically, closure under linear combinations of finite length ensures that S is a subspace.

The given statement leverages this idea, emphasizing the closure under vectors of the form $u + c v$, which can generate the other operations needed.

Step-by-Step Proof of the Statement

Our goal is to demonstrate that the condition:

> For all $u, v \in S$ and all $c \in F$, $u + c v \in S$,

implies that S is a subspace of V.

Step 1: Verify S is nonempty

Given in the statement: S is a nonempty subset of V. Let $u_0 \in S$.

Step 2: Show S contains the zero vector

To verify subspace criteria, we need to confirm that S contains the zero vector $0 \in V$.

- Choose any $u \in S$.
- Set $v = u$ and $c = -1$.
- Then, $u + (-1) \cdot u = u - u = 0$.
- Since the condition holds for all $u, v \in S$, it follows that:

$$
u + (-1) \cdot u = 0 \in S.
$$

Thus, $0 \in S$.

Step 3: Show S is closed under vector addition

- Take any $u, v \in S$.
- Our aim: prove $u + v \in S$.

- Use the given property:

$$
u + 1 \cdot v = u + v \in S,
$$

since $c = 1$ and both u, v are in S.

- Therefore, S is closed under addition.

Step 4: Show S is closed under scalar multiplication

- Take any $u \in S$ and scalar $c \in F$.
- To show $c u \in S$, consider the following:

- We already have that for any $u, v \in S$, $u + c v \in S$.
- To connect this with scalar multiplication, set $v = 0$ (which is in S, as shown previously), and $u \in S$.

- For scalar c, the property gives:

$$
u + c \cdot 0 = u + 0 = u \in S,
$$

which doesn't directly help. Instead, consider the following:

- Since $0 \in S$, for any $u \in S$ and scalar c, set:

$$
u + c u = (1 + c) u,
$$

which is in S by the property, because $u, u \in S$. But this only gives us that:

$$
(1 + c) u \in S,
$$

not necessarily $c u$.

To derive $c u \in S$, we use the following approach:

- Recall that for any $u, v \in S$ and scalar c, the given property states:

$$
u + c v \in S.
$$

- Now, set $v = u$, so that:

$$
u + c u = (1 + c) u \in S.
$$

- Since $(1 + c) u \in S$, and S contains 0, we can write:

$$
(1 + c) u = u + c u,
$$

and by subtracting u (which is in S) from both sides (using the fact that S contains the additive inverse of u, which we will verify next), we can attempt to isolate $c u$.

However, our current steps do not explicitly confirm the closure under scalar multiplication. To complete this, we need to demonstrate that for all $c \in F$, $c u \in S$.

Alternative approach:

- Since $0 \in S$, and for any $u \in S$:

$$
u + c \cdot 0 = u \in S,
$$

which is trivial. But this does not directly give closure under scalar multiplication.

Key insight:

- The given condition $u + c v \in S$ for all $u, v \in S$ and all $c \in F$, combined with the fact that $0 \in S$, suggests a more general approach:

$$
\text{For any } u \in S, \text{ and any } c \in F,

\]
\[
\text{Set } v = u, \text{ then } u + c u \in S.
\]
But, as earlier, this gives \((1 + c) u \in S \).

- Now, observe that \(c u = (1 + c) u - u \).

- Since \((1 + c) u \in S \) and \(u \in S \), their difference:

\[
[(1 + c) u] - u = c u \in S,
\]

if \(

Frequently Asked Questions

What are the necessary conditions for a subset S of a vector space V to be considered a subspace?

A subset S of a vector space V is a subspace if it is nonempty, closed under vector addition, and closed under scalar multiplication. That is, for any u, v ∈ S and any scalar c, both u + v and cu are in S.

How does one prove that a subset S of a vector space V is a subspace?

To prove S is a subspace, show that S is nonempty (often by verifying it contains the zero vector), and then demonstrate that for any u, v ∈ S and scalar c, both u + v and cu are also in S, satisfying closure properties.

Why is the nonempty condition important when proving S is a subspace?

Because the presence of at least one vector in S (commonly the zero vector) ensures the subset is not empty, which is a prerequisite for being a subspace. It also helps establish the zero vector's inclusion, which is essential for subspace properties.

Can a subset of a vector space be a subspace if it contains the zero vector but is not closed under addition?

No, containing the zero vector alone is not sufficient. The subset must also be closed under addition and scalar multiplication. If it isn’t closed under addition, it cannot be a subspace.

What is the significance of closure under scalar multiplication in the context of subspaces?

Closure under scalar multiplication ensures that multiplying any vector in the subset by any scalar results in a vector still within the subset, maintaining the structure necessary for vector space properties within the subset.

How does the proof of the statement in the problem relate to the concepts of subspace criteria?

The proof involves verifying that the subset S satisfies the three subspace criteria: nonemptiness, closure under addition, and closure under scalar multiplication, thereby establishing S as a subspace of V.

What is a common approach to prove that a specific subset S of V is a subspace?

A common approach is to first show S is nonempty (e.g., contains the zero vector), then prove closure under addition and scalar multiplication for arbitrary vectors in S, thus confirming all subspace conditions.

Additional Resources

Vector Space Subset Theorem: An In-Depth Exploration

Introduction: Understanding the Foundations of Vector Spaces

In the realm of linear algebra, the concept of vector spaces forms the bedrock of numerous mathematical and applied disciplines—from computer graphics and engineering to quantum mechanics and data science. At their core, vector spaces are sets equipped with operations of addition and scalar multiplication that satisfy specific axioms, allowing for the meaningful manipulation of vectors in an abstract yet highly structured environment.

One critical aspect of working within vector spaces involves understanding their subsets and the conditions under which these subsets themselves inherit the structure of a vector space. This leads us directly to the focal point of this discussion: the properties and proofs concerning nonempty subsets of a vector space and their status as subspaces.

Understanding the Statement: The Core of the Proposition

The proposition we're examining is foundational in linear algebra:

> Let V be a vector space, and S be a nonempty subset of V. To determine whether S is a subspace of V,
> it suffices to verify that S is closed under addition and scalar multiplication.

More formally, the statement can be summarized as:

> If S is a subset of V such that:
> 1. S is nonempty,
> 2. For every u, v in S, the sum $u + v$ is also in S,
> 3. For every v in S and scalar c, the scalar multiple $c v$ is in S,
> then S is a subspace of V.

This criterion provides a practical and elegant shortcut for mathematicians and students alike to identify subspaces without verifying all of the axioms of vector spaces for the subset, since these properties are inherited under the given conditions.

Why Is This Important? The Significance of the Theorem

Understanding when a subset of a vector space is itself a vector space—a subspace—is crucial for numerous mathematical operations. Subspaces serve as the building blocks for constructing bases, understanding dimensions, and decomposing vector spaces into simpler parts.

The importance of the theorem stems from:

- Simplification: Instead of verifying all vector space axioms (which can be tedious), it reduces the problem to checking closure properties.
- Foundation for Subspace Theory: It underpins many advanced topics like quotient spaces, direct sums, and orthogonal complements.
- Practical Applications: In applied fields, identifying subspaces allows for efficient data encoding, signal processing, and solving linear systems.

Breaking Down the Proof: Step-by-Step Analysis

The proof of the statement hinges on demonstrating that the subset S satisfies all the axioms of a vector space, given the closure properties and nonemptiness.

1. Nonemptiness and the Zero Vector

Since S is nonempty, there exists at least one vector in S. To establish S as a subspace, we need to verify that the zero vector of V is in S.

- Key Point: The zero vector, 0, must be in S for S to be a subspace.
- Method: Use the closure under scalar multiplication by zero.

Proof Step:
Select any vector v in S (which exists because S is nonempty).
Since S is closed under scalar multiplication,
\[0 \times v = 0 \] (the zero vector in V).
Because scalar multiplication of any vector in S by 0 yields the zero vector, and scalar multiplication preserves S's closure, 0 must be in S.

Conclusion: The zero vector 0 belongs to S.

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2. Closure Under Addition

Given that for any u, v in S, the sum u + v is also in S.
This property ensures that S is closed under addition.

- Implication: Addition within S doesn't produce elements outside of S.
- Significance: This is one of the core axioms of vector spaces, ensuring linear combinations are well-defined and contained within the subset.

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3. Closure Under Scalar Multiplication

Similarly, for any v in S and scalar c, the product c v is in S.

- Implication: Scaling vectors in S by any scalar keeps the result within S.
- Significance: This ensures that the linear structure is preserved within the subset.

Completing the Proof: Verifying the Vector Space Axioms

Having established nonemptiness, and closure under addition and scalar multiplication, we now verify that S satisfies all vector space axioms:

Axiom	Explanation	How It's Satisfied in S
Associativity of addition	For u, v, w in S, (u + v) + w = u + (v + w)	Holds because S inherits associativity from V
Commutativity of addition	For u, v in S, u + v = v + u	Holds for the same reason
Existence of additive identity	Exists as 0 in S (proved earlier)	Zero vector is in S
Existence of additive inverses	For v in S, -v in S	Since v in S, and scalar multiplication by -1 (which is in the field) preserves S, -v is in S
Closure under scalar multiplication	Already given	By assumption
Distributivity, scalar identity, etc.	Inherited from V	Due to the structure of V

Result: All axioms are satisfied, and thus S is a subspace of V.

Practical Implications and Applications

The theorem's elegance lies in its simplicity and utility, especially when examining complex vector spaces in applied mathematics.

Applications include:

- Subspace Identification: Given a data set, verify if it forms a subspace by checking closure properties.
- Subspace Construction: Building subspaces by combining known vectors and verifying closure.
- Solution Spaces: The set of solutions to a homogeneous linear system forms a subspace, verified via these closure properties.
- Functional Analysis: When working with function spaces, subsets like the set of continuous functions,

differentiable functions, etc., can be checked for subspace status through these criteria.

Limitations and Considerations

While the theorem provides a straightforward method, some caveats include:

- Nonemptiness is essential: A subset that is empty cannot be a subspace because it lacks the zero vector.
- Closure properties are necessary but not sufficient alone: In some cases, verifying closure is straightforward, but care must be taken to ensure the subset is nonempty to meet all criteria.
- Subspace verification in infinite-dimensional spaces: The same principles apply, but checking closure might involve more intricate functional analysis.

Summary: The Power of the Subspace Criterion

In conclusion, the statement offers a powerful and efficient way to determine whether a subset of a vector space qualifies as a subspace. By ensuring the subset is nonempty and closed under addition and scalar multiplication, one guarantees that it inherits the full structure of a vector space. This result simplifies analysis and underpins much of linear algebra's theoretical framework and practical applications.

Remember: At its heart, the theorem emphasizes the importance of closure properties and the presence of the zero vector, anchoring the concept of subspaces in the foundational axioms of vector spaces. Whether you're solving systems of linear equations, analyzing function spaces, or working with data in high-dimensional spaces, this criterion remains a fundamental tool in your mathematical toolkit.

In essence, understanding and applying this theorem unlocks the ability to explore the rich landscape of vector spaces methodically, systematically, and confidently.

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