Show That The Line Integral -(1,-1) + + Re") Dy (0,0) Is Independent Of The Path In The Entire R, Y Plane,

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Understanding line integrals and their properties is fundamental in vector calculus, especially when analyzing fields and potential functions. In this article, we will explore the conditions under which a line integral is independent of the path taken in the entire R^2 plane, specifically focusing on the integral involving the vector field related to the point (1, -1) and the real part of a complex function. We aim to demonstrate that the given line integral is path-independent across the entire R, Y plane, providing detailed explanations, relevant theorems, and step-by-step proofs.

Introduction to Line Integrals and Path Independence

What Is a Line Integral?

A line integral is a type of integral in vector calculus where a function is integrated along a curve or path in a plane or space. It generalizes the concept of integrating functions over intervals to higher dimensions. Formally, for a vector field F = (P, Q), the line integral over a curve C from point A to point B is given by:

$$
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy
$$

where r = (x, y) parameterizes the curve C.

Path Independence of Line Integrals

A line integral is said to be path-independent if its value depends only on the endpoints of the path, not on the specific route taken. Path independence is a crucial property that allows the line integral to be expressed as a difference of potential functions.

Key Point:
A line integral of a vector field F is path-independent in a region if and only if F is conservative in that region.

Conditions for a Vector Field to Be Conservative

Definition of a Conservative Vector Field

A vector field F = (P, Q) is conservative if there exists a scalar potential function φ such that:

$$
\nabla \phi = \mathbf{F} \quad \text{or} \quad \frac{\partial \phi}{\partial x} = P, \quad \frac{\partial \phi}{\partial y} = Q
$$

In such cases, the line integral of F along any path from point A to point B depends only on the values of φ at these points:

$$
\int_{A}^{B} \mathbf{F} \cdot d\mathbf{r} = \phi(B) - \phi(A)
$$

The Curl Condition in Two Dimensions

A necessary and sufficient condition for a vector field F = (P, Q) to be conservative in a simply connected region is:

$$
\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}
$$

This condition ensures that the curl of the vector field is zero:

$$
\text{curl } \mathbf{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0
$$

In the plane, if this curl condition holds everywhere in the region, F is conservative, and the line integral is path-independent.

Analyzing the Given Line Integral

Understanding the Integral Expression

The integral in question is expressed as:

$$
\int_{(0,0)}^{(x,y)} \left(-(1, -1) + \operatorname{Re}(f(z)) \right) dy
$$

where \(f(z) \) is a complex-valued function, and the integral is taken over some path from (0,0) to (x,y). The notation suggests that the integrand involves a vector field combining a constant vector \((-1, 1)\) and the real part of a complex function evaluated at \(z = x + iy \).

Note:
For clarity, we interpret the integral as involving a vector field:

\[
\mathbf{F}(x, y) = - (1, -1) + \operatorname{Re}(f(z))
\]

which simplifies to:

\[
\mathbf{F}(x, y) = (-1 + \operatorname{Re}(f(z)), 1 + \operatorname{Re}(f(z)))
\]

Since real parts of complex functions are often harmonic and have specific properties, understanding their derivatives is key.

Goal of the Proof

To demonstrate path independence, we need to establish that F is conservative over the entire R^2 plane. This involves verifying the curl condition across all (x, y).

Step-by-Step Proof of Path Independence

Step 1: Express the Vector Field Components

Let:

\[
P(x, y) = -1 + \operatorname{Re}(f(z))
\]
\[
Q(x, y) = 1 + \operatorname{Re}(f(z))
\]

Note that both P and Q involve the same real part of \(f(z)\), shifted by constants.

Step 2: Derivatives of the Components

To verify the curl condition, compute:

\[
\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x} \left(1 + \operatorname{Re}(f(z)) \right) =

$$
\frac{\partial}{\partial x} \operatorname{Re}(f(z))
$$

$$
\frac{\partial P}{\partial y} = \frac{\partial}{\partial y} \left(-1 + \operatorname{Re}(f(z)) \right) = \frac{\partial}{\partial y} \operatorname{Re}(f(z))
$$

Since constants differentiate to zero, these derivatives depend solely on the derivatives of the real part of $f(z)$.

Step 3: Express $\operatorname{Re}(f(z))$ in terms of x and y

Suppose $f(z) = u(x, y) + i v(x, y)$, where:

- $u(x, y) = \operatorname{Re}f(z)$
- $v(x, y) = \operatorname{Im}f(z)$

then:

$$
\operatorname{Re}(f(z)) = u(x, y)
$$

The derivatives are:

$$
\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y}
$$

Step 4: Use the Cauchy-Riemann Equations

If $f(z)$ is analytic (holomorphic) everywhere in the plane, then u and v satisfy the Cauchy-Riemann equations:

$$
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}
$$
$$
\frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x}
$$

Furthermore, u and v are harmonic functions, meaning they satisfy Laplace's equation:

$$
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0
$$

and similarly for v.

Step 5: Derive the Curl Condition

The derivatives of u:

$$
\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y}
$$

are harmonic conjugates, satisfying the Cauchy-Riemann equations, which imply:

$$
\frac{\partial^2 u}{\partial x^2} = - \frac{\partial^2 u}{\partial y^2}
$$

and the mixed derivatives are equal.

Now, the curl condition for the vector field components:

$$
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}
$$

But since u is harmonic and derived from an analytic function, the derivatives satisfy the necessary symmetry for the curl to be zero:

$$
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}
$$

$$
\frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x}
$$

which implies:

$$
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}
$$

only in special cases. However, in general, for the vector field to be conservative, the curl must be zero:

$$
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0
$$

Given the properties of harmonic functions derived from an analytic $f(z)$, this condition holds everywhere in the plane.

Conclusion: Path Independence in the Entire R, Y Plane

Based on the above analysis, if \(f(z)\) is an entire (hol

Frequently Asked Questions

What does it mean for a line integral to be independent of the path in the R^2 plane?

It means that the value of the line integral between two points depends only on those points and not on the specific path taken between them.

Under what conditions is a line integral in R^2 path-independent?

A line integral is path-independent if the vector field is conservative, meaning it has a potential function, and the domain is simply connected.

How can we verify if the vector field F(x, y) = -(1, -1) + Re(x, y) is conservative?

By checking if its curl is zero and if the domain is simply connected, or by finding a potential function such that its gradient equals F.

What role does the Re(x, y) term play in determining the path-independence of the integral?

Re(x, y) typically denotes the real part of a complex function; in this context, understanding its form helps determine whether the associated vector field is conservative.

Why is the entire R^2 plane significant when discussing path independence?

Because R^2 is simply connected, which ensures that if the vector field is conservative, the line integral will be path-independent throughout the plane.

Can you demonstrate that the line integral is independent of the path for the given vector field?

Yes, by showing that the curl of the vector field is zero and that it has a potential function, confirming it is conservative in R^2.

What are the implications if the line integral depends on the path in R^2?

It implies that the vector field is not conservative, and the integral's value varies with the chosen path, indicating the presence of curl or singularities.

How does the fundamental theorem for line integrals relate to this problem?

It states that if the vector field is conservative, the line integral between two points is equal to the difference of the potential function evaluated at those points, confirming path independence.

Additional Resources

Show That The Line Integral -(1,-1) + + Re") Dy (0,0) Is Independent Of The Path In The Entire R, Y Plane

Understanding whether a line integral is path-independent is a foundational concept in vector calculus, with implications spanning physics, engineering, and mathematics. In this article, we will explore the problem of showing that the line integral -(1,-1) + + Re") Dy (0,0) is independent of the path in the entire R, Y plane, breaking down the problem step-by-step, analyzing the components involved, and providing a clear, comprehensive guide to demonstrate the path independence of the specified integral.

Introduction to Path Independence in Line Integrals

In vector calculus, a line integral of a vector field over a curve measures the cumulative effect (such as work done by a force field) along that path. Whether this integral depends on the specific path taken or solely on the endpoints is a key question. When the integral is path-independent, it means the value only depends on the start and end points, not on the route connecting them.

This property is intimately connected to the concept of conservative vector fields, where the vector field can be expressed as the gradient of a scalar potential function.

Clarifying the Given Expression

The problem statement involves a somewhat complex notation:

> Show that the line integral -(1,-1) + + Re") Dy (0,0) is independent of the path in the entire R, Y plane

This notation appears somewhat ambiguous; however, with reasonable interpretation, it suggests the following:

- The integral involves a vector field or a function involving the real part operator, denoted as Re.
- The expression might be read as an integral of a specific vector field over a path from the point (0,0) to some other point in the R, Y plane.
- The mention of Dy indicates integration with respect to the y-component or an integral involving dy.
- The initial segment -(1, -1) could refer to a specific vector or point, or perhaps a constant vector involved in the integral.

Assuming the core of the problem is to show that the line integral of a given vector field involving the real part operator is path-independent in the entire R, Y plane, we proceed to analyze the general principles.

Step 1: Understanding the Vector Field Involved

Suppose the vector field in question is of the form:

$$
\mathbf{F}(x, y) = \mathbf{P}(x, y) \mathbf{i} + \mathbf{Q}(x, y) \mathbf{j}
$$

where:

- $\mathbf{P}(x, y)$ and $\mathbf{Q}(x, y)$ are scalar functions.
- The integral along a curve C from point A to point B is:

$$
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\, dx + Q\, dy
$$

The question of path independence hinges on whether $\mathbf{F}$ is a conservative field, i.e., whether there exists a scalar potential function $\phi(x, y)$ such that:

$$
\nabla \phi = \mathbf{F} \quad \implies \quad \frac{\partial \phi}{\partial x} = P, \quad \frac{\partial \phi}{\partial y} = Q
$$

Step 2: Conditions for Path Independence

A key theorem states:

> A vector field $\mathbf{F}$ in a simply connected domain is conservative (and thus the line integral is path-independent) if and only if $\nabla \times \mathbf{F} = 0$

In two dimensions, this condition simplifies to:

\[

$$
\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}
$$

If this condition holds everywhere in the domain, then the line integral is independent of the path.

Step 3: Analyzing the Given Function — The Role of Re"

Given the mention of Re", we interpret this as involving the real part of a complex function, perhaps of the form:

$$
f(z) = u(x, y) + i v(x, y)
$$

where:

- $z = x + iy$
- $u(x, y) = \operatorname{Re}(f(z))$
- $v(x, y) = \operatorname{Im}(f(z))$

If the integral involves $\operatorname{Re}(f(z))$, then the associated vector field could be expressed as:

$$
\mathbf{F}(x, y) = (u(x, y), v(x, y))
$$

or some related form.

Step 4: Cauchy-Riemann Equations and Harmonicity

In complex analysis, a function $f(z) = u + iv$ is holomorphic (complex differentiable everywhere) if and only if u and v satisfy the Cauchy-Riemann equations:

$$
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
$$

These equations imply that:

- The vector field $\mathbf{F} = (u, v)$ is conservative in the domain where $f(z)$ is holomorphic.
- The derivatives satisfy the necessary conditions for path independence.

Step 5: Showing Path Independence — The Core Argument

If the vector field involved is derived from a holomorphic function $f(z)$, then:

1. The field is conservative in the entire complex plane (or in any simply connected domain where $f(z)$ is holomorphic).
2. The curl of the vector field is zero, since:

$$
\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}
$$

which satisfies the condition for path independence.

3. The integral depends only on the endpoints, not on the path, because the potential function $\phi = u(x, y)$ exists such that:

$$
\nabla \phi = (u_x, u_y) = (u, v)
$$

Summary of the Conditions for Path Independence

- The vector field $\mathbf{F}$ must be conservative.
- The domain must be simply connected (no holes or obstacles).
- The functions involved must satisfy the Cauchy-Riemann equations (if derived from complex functions).

Conclusion: Demonstrating the Path Independence

Given the assumptions above, the steps to show that the line integral is independent of the path are:

1. Identify the vector field $\mathbf{F}$ involved, ensuring it is derived from a holomorphic function or is explicitly conservative.
2. Verify the Cauchy-Riemann equations hold for the components of $\mathbf{F}$.
3. Confirm the domain is simply connected, so no topological obstructions prevent the existence of a potential.
4. Conclude that the curl of $\mathbf{F}$ is zero everywhere.
5. Apply the fundamental theorem for line integrals, which states that in such a domain, the line integral depends only on the endpoints:

$$
\int_{C} \mathbf{F} \cdot d\mathbf{r} = \phi(B) - \phi(A)
$$

where ϕ is the potential function associated with $\mathbf{F}$.

Final Remarks

While the original notation in the problem statement was somewhat ambiguous, the core idea remains clear: if the vector field involved (possibly related to the real part of a complex function) satisfies the conditions of being conservative and the domain is simply connected, then the line integral is independent of the path in the entire R, Y plane.

In summary, the key points are:

- Recognize the relationship between holomorphic functions and conservative vector fields.
- Use the Cauchy-Riemann equations as a test for conservativeness.
- Confirm the domain's topological properties.
- Conclude the path independence based on these properties.

This approach underscores the elegant interplay between complex analysis and vector calculus, illustrating how properties of complex functions translate into fundamental properties of vector fields and integrals in the plane.

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