Use The Binomial Series To Expand The Function As A Power Series. Find The Radius Of Convergence. Question

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Understanding how to expand functions into power series is a fundamental aspect of mathematical analysis, particularly in calculus and complex analysis. One of the most powerful tools for such expansions is the binomial series, which allows us to express functions of the form $(1 + x)^n$ as an infinite series. This technique not only simplifies complex functions but also provides insights into their behavior near specific points, especially around the point $x = 0$.

In this comprehensive guide, we will delve into the process of using the binomial series to expand functions into power series. We will also explore how to determine the radius of convergence of these series, which is crucial for understanding the interval within which the series accurately represents the function. This discussion is framed around a typical exam or textbook question that tests your ability to apply binomial expansion techniques and analyze convergence properties.

Understanding the Binomial Series

What is the Binomial Series?

The binomial series is an infinite series expansion of the expression $(1 + x)^n$, where n can be any real number (not necessarily an integer). The general binomial series is expressed as:

$$
(1 + x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k
$$

where $\binom{n}{k}$ is the binomial coefficient given by:

$$
\binom{n}{k} = \frac{n(n-1)(n-2) \cdots (n - k + 1)}{k!}
$$

for $k \geq 1$, and $\binom{n}{0} = 1$.

This series converges for $|x| < 1$ when n is not a non-negative integer, and converges for all x if n is a non-negative integer (since then the series terminates).

Applications of the Binomial Series

The binomial series is useful in:
- Expanding functions like $(1 + x)^n$ for non-integer n,
- Deriving approximations of functions near specific points,
- Solving differential equations,
- Analyzing series convergence,
- Simplifying complex algebraic expressions in calculus.

Expanding a Function Using the Binomial Series

Suppose you are asked to expand a function $f(x)$ into a power series around $x=0$. The process generally involves:

1. Expressing the function in a form suitable for binomial expansion.
2. Applying the binomial theorem to generate the series.
3. Simplifying the resulting expression to identify the power series.

Let's consider a typical example:

Example: Expand $\sqrt{1 + x}$ as a power series using the binomial series.

Step 1: Recognize the form

Note that:

$$
\sqrt{1 + x} = (1 + x)^{1/2}
$$

which matches the binomial form with $n = 1/2$.

Step 2: Write the binomial series

Using the binomial expansion:

$$
(1 + x)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} x^k

\]

where:

\[
\binom{1/2}{k} = \frac{\frac{1}{2} (\frac{1}{2} - 1) (\frac{1}{2} - 2) \cdots (\frac{1}{2} - k + 1)}{k!}
\]

Step 3: Calculate the first few terms

- For \(k=0\):

\[
\binom{1/2}{0} = 1
\]

- For \(k=1\):

\[
\binom{1/2}{1} = \frac{1/2}{1} = \frac{1}{2}
\]

- For \(k=2\):

\[
\binom{1/2}{2} = \frac{\frac{1}{2} \left(\frac{1}{2} - 1\right)}{2!} = \frac{\frac{1}{2} \times -\frac{1}{2}}{2} = -\frac{1}{8}
\]

- For \(k=3\):

\[
\binom{1/2}{3} = \frac{\frac{1}{2} \times -\frac{1}{2} \times -\frac{3}{2}}{6} = \frac{\frac{1}{2} \times -\frac{1}{2} \times -\frac{3}{2}}{6} = \frac{\frac{3}{8}}{6} = \frac{1}{16}
\]

Thus, the power series expansion up to the first few terms is:

\[
\sqrt{1 + x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots
\]

Determining the Radius of Convergence

Knowing the power series expansion alone isn't sufficient; understanding where the series converges (i.e., the radius of convergence) is equally important. The radius of convergence determines the interval in which the power series accurately represents the function.

Method 1: Using the Root or Ratio Test

The most common approach for finding the radius of convergence is applying the Ratio Test or the Root Test to the terms of the series.

Ratio Test: For a series $\sum a_k x^k$, the series converges when:

$$
\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| < 1
$$

Applying this to the binomial series:

$$
a_k = \binom{n}{k}
$$

which involves factorials and products, so the ratio test simplifies to:

$$
\lim_{k \to \infty} \left| \frac{\binom{n}{k+1}}{\binom{n}{k}} \right| =
\lim_{k \to \infty} \left| \frac{n - k}{k + 1} \right|
$$

This limit tends to zero for all finite n when $k \to \infty$, which indicates the series converges for:

$$
|x| < 1
$$

Hence, the radius of convergence is:

$$
R = 1
$$

Note: For series with rational exponents (like $\sqrt{1 + x}$), the binomial series converges for $|x| < 1$. The behavior at $|x|=1$ (boundary points) needs separate analysis.

Method 2: Recognizing the Series as a Function of $(1 + x)$

Since the binomial series expansion for $(1 + x)^n$ converges for $|x| < 1$, this radius applies generally unless the function's domain limits convergence differently.

Summary of the Process

To effectively use the binomial series for expansion and determine the radius of convergence, follow these steps:

1. **Identify the function:** Express the function in the form $(1 + x)^n$ or manipulate it to this form.

2. **Apply the binomial theorem:** Write the series expansion using the binomial coefficients.

3. **Compute initial terms (if needed):** Calculate the first few terms explicitly for approximation or insight.

4. **Determine the radius of convergence:** Use the Ratio Test or Root Test to analyze the limit of series terms as $k \to \infty$.

5. **Assess convergence at boundary points:** Examine the series at $|x| = 1$ separately to establish convergence or divergence.

Practical Applications and Examples

Example 1: Expand $(1 - 2x)^{3/2}$ as a power series

- Recognize the form: $n=3/2$, x replaced by $-2x$.
- Write the series:

$$
(1 - 2x)^{3/2} = \sum_{k=0}^{\infty} \binom{3/2}{k} (-2x)^k
$$

- Find the binomial coefficients:

$$
\binom{3/2}{k} = \frac{\frac{3}{2} \left(\frac{3}{2} - 1\right) \left(\frac{3}{2} - 2\right) \cdots}{k!}
$$

- Calculate the first few terms to approximate the function

Frequently Asked Questions

How do you use the binomial series to expand a function like $(1 + x)^n$ as a power series?

To expand $(1 + x)^n$ using the binomial series, you write it as an infinite sum: $(1 + x)^n = \Sigma$ (n choose k) x^k, where (n choose k) $= n(n-1)(n-2)...(n-k+1)/k!$. This series converges for $|x| < 1$ when n is not a positive integer.

What is the general form of the binomial series expansion for $(1 + x)^n$?

The binomial series expansion is $(1 + x)^n = \Sigma_{k=0}^\infty$ (n choose k) x^k, where (n choose k) $= n(n-1)(n-2)...(n-k+1)/k!$ for real or complex n, and the series converges for $|x| < 1$.

How do you find the radius of convergence for the binomial series expansion?

For the binomial series expansion of $(1 + x)^n$ with real n, the radius of convergence is $|x| < 1$. This is because the binomial series is a power series centered at 0 and converges within the unit circle, unless n is a non-negative integer where the series terminates.

Can the binomial series be used to expand functions with negative or fractional exponents?

Yes, the binomial series can be used to expand functions with negative or fractional exponents, such as $(1 + x)^{-k}$ or $(1 + x)^{p}$, where p is any real number, provided $|x| < 1$ for convergence.

What is the significance of the radius of convergence when expanding functions as a power series?

The radius of convergence determines the interval around the expansion point within which the power series converges to the function. For the binomial series, it indicates the range of x-values for which the expansion accurately represents the function.

How do you determine the radius of convergence for a binomial series expansion in practice?

In practice, for the binomial series $(1 + x)^n$, the radius of convergence is typically 1, since the series converges for $|x| < 1$. To confirm, you can apply the ratio test or root test to the general term of the series, which usually yields a radius of 1 unless the series terminates.

Additional Resources

Use The Binomial Series To Expand The Function As A Power Series. Find The Radius Of Convergence. Question

The binomial series is a fundamental tool in calculus and mathematical analysis that allows us to expand functions of the form $(1 + x)^{\alpha}$ into an infinite power series. This powerful technique is especially useful for approximating functions near a specific point, analyzing their behavior, and solving complex problems where direct computation is challenging. In this guide, we will explore how to use the binomial series to expand a given function as a power series and determine its radius of convergence, providing a comprehensive understanding of the process, underlying principles, and applications.

Understanding the Binomial Series

What is the Binomial Series?

The binomial series extends the binomial theorem, which typically applies to positive integers, to include any real (or complex) exponent α. The general form of the binomial series expansion for $(1 + x)^{\alpha}$ is:

$$
(1 + x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^{n}
$$

where the binomial coefficient $\binom{\alpha}{n}$ is defined as:

$$
\binom{\alpha}{n} = \frac{\alpha (\alpha - 1) (\alpha - 2) \cdots (\alpha - n + 1)}{n!}
$$

for $n \geq 1$, with $\binom{\alpha}{0} = 1$.

Conditions for Validity

- The series converges for $|x| < 1$ when α is not a non-negative integer.
- For $\alpha \in \mathbb{N}$, the series terminates after $n = \alpha$, becoming a finite polynomial.
- The convergence at $|x| = 1$ depends on the specific value of α.

Significance in Analysis

Using the binomial series provides a way to approximate functions near points where they are difficult to evaluate directly, especially around $x=0$. It is invaluable in calculus, differential equations, and mathematical modeling, allowing for series solutions, asymptotic analysis, and numerical approximations.

Step-by-Step Guide to Using the Binomial Series to Expand a Function

Suppose you are asked to expand a function of the form $f(x) = (1 + g(x))^{\alpha}$ as a power series centered at $x=0$. The process involves several key steps:

1. Recognize the Function’s Form

Identify whether the function can be expressed in the form $(1 + x)^{\alpha}$. If not, manipulate it algebraically to achieve this form.

Example: To expand $f(x) = (1 + 2x)^{\alpha}$, note that:

$$
f(x) = \left(1 + 2x\right)^{\alpha}
$$

which can be written as $(1 + x)^{\alpha}$ with x replaced by $2x$.

2. Express the Function in the Binomial Series Format

Rewrite the function in a form suitable for applying the binomial theorem:

$$
f(x) = (1 + h(x))^{\alpha}
$$

where $h(x)$ is a function of x that is small near the expansion point (usually $x=0$).

3. Expand Using the Binomial Series

Apply the binomial series expansion:

$$
(1 + h(x))^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} [h(x)]^{n}
$$

Compute the binomial coefficients and raise $h(x)$ to the n-th power.

Example: For $f(x) = (1 + 2x)^{\alpha}$:

$$
f(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} (2x)^{n}
$$

which simplifies to:

$$
f(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} 2^{n} x^{n}
$$

4. Write Out the First Few Terms

To get a practical approximation or to analyze behavior, write out the terms explicitly:

$$
f(x) \approx \binom{\alpha}{0} + \binom{\alpha}{1} 2 x + \binom{\alpha}{2} 2^{2} x^{2} + \cdots
$$

Knowing the coefficients explicitly helps in understanding the function’s behavior near $x=0$.

5. Determine the Radius of Convergence

The radius of convergence tells us the interval around the expansion point where the power series converges to the actual function. To find it:

- Use the ratio test or root test on the general term.
- Recognize that the series converges for $|h(x)| < 1$.

Since the binomial series converges when $|h(x)| < 1$, the radius of convergence R is given by solving:

$$
|h(x)| < 1
$$

for x.

Calculating the Radius of Convergence

1. Applying the Ratio Test

The ratio test states that a series $\sum a_{n}$ converges if:

$$
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_{n}} \right| < 1
$$

For the binomial series:

$$
a_{n} = \binom{\alpha}{n} [h(x)]^{n}
$$

we analyze the behavior of $\binom{\alpha}{n}$ as $n \to \infty$. Using properties of the binomial coefficients, the dominant factor is:

$$
\left| \frac{\binom{\alpha}{n+1}}{\binom{\alpha}{n}} \right| \approx \frac{|\alpha - n|}{n+1}
$$

which tends to 1 as $n \to \infty$. Therefore, the convergence depends mainly on $|h(x)|$.

2. Convergence Condition

The binomial series converges when:

$$
|h(x)| < 1
$$

In practice, this means the radius of convergence in terms of x is determined by the magnitude of $h(x)$:

$$
|h(x)| < 1
$$

For example, if $h(x) = c x$, then:

$$
|c x| < 1 \implies |x| < \frac{1}{|c|}
$$

Similarly, for more complex $h(x)$, solve the inequality accordingly.

3. Example: Expanding $(1 + 3x)^{\alpha}$

Here, $h(x) = 3x$:

$$
|3x| < 1 \implies |x| < \frac{1}{3}
$$

Thus, the radius of convergence is $R = \frac{1}{3}$.

Practical Examples

Example 1: Expand $(1 + x)^{\frac{1}{2}}$

Solution:

- Recognize $\alpha = \frac{1}{2}$, $h(x) = x$.
- Expand using the binomial series:

$$
(1 + x)^{1/2} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^{n}
$$

- Binomial coefficients:

$$
\binom{\frac{1}{2}}{n} = \frac{\frac{1}{2} (\frac{1}{2} - 1) (\frac{1}{2} - 2) \cdots (\frac{1}{2} - n + 1)}{n!}
$$

- First few terms:

$$
(1 + x)^{1/2} \approx 1 + \frac{1}{2} x - \frac{1}{8} x^{2} + \frac{1}{16} x^{3} - \cdots
$$

- Radius of convergence: $|x| < 1$.

Example 2: Expand $(1 - 4x)^{3/2}$

Solution:

- Recognize $h(x) = -4x$, $\alpha = \frac{3}{2}$.
- Series expansion:

$$
(1 - 4x)^{3/2} = \sum_{n=0}^{\infty} \binom{\frac{3}{2}}{n} (-4x)^{n}
$$

- Radius of convergence:

\[

```
| -4x | < 1 \implies |x| < \frac{1}{4}
\]

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Summary and Key Takeaways

- The binomial series provides a method to expand functions of the form \((1
+ x)^{\alpha}\) into an infinite power series.
- To utilize the binomial series effectively:
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