Verify That Each Given Function Is A Solution Of The Differential Equation. 5. Y"-y=0; Y_1(t) = E', Y_2(t)

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Understanding how to verify whether a given function is a solution to a differential equation is fundamental in differential equations. In this article, we will explore the process of verifying solutions for the specific second-order differential equation:

$$ Y'' - y = 0 $$

with particular functions $Y_1(t) = e^t$ and $Y_2(t) = e^{-t}$. We will systematically analyze each function, perform the necessary derivatives, and verify whether they satisfy the differential equation. This process is crucial for students and practitioners working with differential equations, as it helps confirm solutions and understand the structure of the solution space.

Understanding the Differential Equation $Y'' - y = 0$

Before verifying solutions, it’s essential to interpret and understand the differential equation itself.

Form of the Differential Equation

The given differential equation:

$$ Y'' - y = 0 $$

is a second-order linear differential equation. It involves the second derivative of the function $Y(t)$ and the function $Y(t)$ itself.

- The term Y'' is the second derivative with respect to t.
- The term Y appears without any coefficient, implying the equation is homogeneous.

Standard Form and Characteristics

Rearranged, the differential equation is:

$$ Y'' = Y $$

This is a classic second-order linear homogeneous differential equation with constant coefficients, where the second derivative equals the function itself.

Methodology for Verifying Solutions

To verify whether a particular function $Y(t)$ is a solution to the differential equation, follow these steps:

1. Compute the derivatives: Find the first derivative Y' and the second derivative Y'' of the function.
2. Substitute into the differential equation: Plug $Y(t)$ and $Y''(t)$ into the left-hand side of the differential equation.
3. Check for equality: Determine if the resulting expression simplifies to zero for all t.

If the substituted expression simplifies to zero identically (for all t in the domain), then $Y(t)$ is a solution to the differential equation.

Verifying $Y_1(t) = e^t$ as a Solution

Step 1: Compute derivatives of $Y_1(t)$

Given:

$$ Y_1(t) = e^t $$

- First derivative:

$$ Y_1'(t) = \frac{d}{dt} e^t = e^t $$

- Second derivative:

$$ Y_1''(t) = \frac{d}{dt} e^t = e^t $$

Step 2: Substitute into the differential equation

Recall the differential equation:

$$ Y'' - y = 0 $$

Substituting $Y_1(t)$ and $Y_1''(t)$:

$$ e^t - e^t = 0 $$

Step 3: Verify the solution

Since:

$$ e^t - e^t = 0 $$

the equation holds true for all t. Therefore:

$Y_1(t) = e^t$ is a solution to the differential equation.

Verifying $Y_2(t) = e^{-t}$ as a Solution

Step 1: Compute derivatives of $Y_2(t)$

Given:

$$ Y_2(t) = e^{-t} $$

- First derivative:

$$ Y_2'(t) = \frac{d}{dt} e^{-t} = -e^{-t} $$

- Second derivative:

$$ Y_2''(t) = \frac{d}{dt} (-e^{-t}) = e^{-t} $$

Step 2: Substitute into the differential equation

Plugging in:

$$Y_2''(t) - Y_2(t) = e^{-t} - e^{-t}$$

3: Verify the solution

Since:

$$e^{-t} - e^{-t} = 0$$

the expression simplifies to zero for all t, confirming that:

$Y_2(t) = e^{-t}$ is also a solution to the differential equation.

Discussion on the General Solution of the Differential Equation

Having verified that both e^t and e^{-t} are solutions, it's important to recognize their significance in the context of the general solution.

The Superposition Principle

Since the differential equation:

$$Y'' - Y = 0$$

is linear and homogeneous, any linear combination of solutions is also a solution:

$$Y(t) = C_1 e^t + C_2 e^{-t}$$

where C_1, C_2 are arbitrary constants.

Basis of the Solution Space

The functions $Y_1(t) = e^t$ and $Y_2(t) = e^{-t}$ form a fundamental set of solutions. They serve as a basis for the solution space of the differential equation.

Additional Methods for Verifying Solutions

While direct substitution is straightforward in this case, other methods can be employed for verifying solutions or solving similar differential equations.

Characteristic Equation Method

For equations with constant coefficients, the characteristic equation provides a systematic approach:

- Convert the differential equation into algebraic form:

$$ r^2 - 1 = 0 $$

- Solve for r:

$$ r^2 = 1 \Rightarrow r = \pm 1 $$

- Write the general solution as:

$$ Y(t) = C_1 e^{t} + C_2 e^{-t} $$

This approach confirms the solutions we verified through direct substitution.

Using the Wronskian

The Wronskian of the two solutions:

$$ W(t) = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = e^t \times (-e^{-t}) - e^t \times e^{-t} = -1 - 1 = -2 \neq 0 $$

Since the Wronskian is non-zero, the solutions are linearly independent, confirming their validity as a fundamental set.

Implications for Differential Equation Solving and Applications

Verifying solutions is a key step in solving differential equations, especially in engineering, physics, and other sciences. Understanding the process ensures accurate modeling of real-world phenomena.

- Physics: Solutions like e^t and e^{-t} often represent exponential growth and decay processes.
- Engineering: These solutions help analyze systems with oscillatory or exponential behaviors.
- Mathematics: Verifying solutions enhances comprehension of linear independence, basis functions, and the structure of solution spaces.

Conclusion

In summary, verifying whether given functions satisfy a differential equation involves calculating derivatives, substituting into the original equation, and confirming the identity holds for all t. For the differential equation $Y'' - Y = 0$, both $Y_1(t) = e^t$ and $Y_2(t) = e^{-t}$ are solutions, forming a fundamental set of solutions. This verification process is a fundamental skill in differential equations, enabling practitioners to confirm solutions, understand the solution space, and apply these solutions effectively in various scientific and engineering contexts.

By mastering these verification techniques, students and professionals can confidently analyze and solve complex differential equations, facilitating advances in modeling, simulation, and problem-solving across numerous fields.

Frequently Asked Questions

How do I verify that a given function Y(t) is a solution to the differential equation Y'' - Y = 0?

To verify, compute the second derivative $Y''(t)$ of the function and check if substituting $Y(t)$ and $Y''(t)$ into the equation results in a true statement (i.e., $Y''(t) - Y(t) = 0$).

Given Y_1(t) = e^t, how can I confirm it is a solution to Y'' - Y = 0?

Calculate $Y_1''(t) = e^t$. Substitute into the differential equation: $e^t - e^t = 0$. Since the equation holds, $Y_1(t) = e^t$ is a solution.

If Y_2(t) = e^{-t}, how do I verify it satisfies the differential equation Y'' - Y = 0?

Compute $Y_2''(t) = e^{-t}$. Substitute into the equation: $e^{-t} - e^{-t} = 0$. Since it holds true, $Y_2(t) = e^{-t}$ is a solution.

What is the general method to determine if a function like Y(t) = c_1 e^t + c_2 e^{-t} is a solution to Y'' - Y = 0?

Substitute Y(t) into the differential equation, compute its derivatives, and verify that the equation simplifies to zero for all t, confirming it is a solution.

Are exponential functions like e^t and e^{-t} always solutions to the differential equation Y'' - Y = 0?

Yes, exponential functions of the form e^{kt} are solutions when k^2 - 1 = 0, i.e., when k = ±1. Therefore, e^t and e^{-t} are solutions to Y'' - Y = 0.

Additional Resources

Verification of Solutions to Differential Equations: An In-Depth Analysis

In the realm of differential equations, the process of verifying whether a given function truly solves a specific equation is both fundamental and essential. This task ensures the correctness of solutions, deepens understanding of the behavior of functions within the context of differential equations, and is a cornerstone skill for mathematicians, engineers, physicists, and anyone working with mathematical modeling. Here, we focus on a particular second-order linear differential equation:

Y'' - y = 0,

and examine how to verify whether provided functions are solutions, specifically considering the functions $Y_1(t)$ = e^t and $Y_2(t)$ = e^{-t}.

Understanding the Differential Equation Y'' - y = 0

Before delving into the verification process, it is crucial to understand the nature of the differential equation in question.

Type and Characteristics

- Order: The differential equation is second-order because the highest derivative involved is the second derivative, denoted as Y''.
- Linearity: It is linear, as the terms involve Y and its derivatives linearly, with no products or powers of Y or its derivatives.
- Homogeneity: The equation is homogeneous because all terms depend on Y or its derivatives, with zero on the right-hand side.

Standard Form and Corresponding Characteristic Equation

The equation can be rewritten as:

$$ Y'' - Y = 0 $$

The typical approach for such an equation involves solving the associated characteristic equation:

$$ r^2 - 1 = 0 $$

which simplifies to:

$$ r^2 = 1 $$

leading to roots:

$$ r = \pm 1 $$

These roots indicate that the general solution to the differential equation is a linear combination of exponential functions:

$$ Y(t) = C_1 e^{t} + C_2 e^{-t} $$

where C_1 and C_2 are arbitrary constants.

Given Functions: $Y_1(t) = e^t$ and $Y_2(t) = e^{-t}$

The focus now shifts to the specific functions provided as potential solutions:

- $Y_1(t) = e^t$
- $Y_2(t) = e^{-t}$

These functions are classic solutions corresponding directly to the roots of the characteristic equation, and verifying whether they satisfy the differential equation is a straightforward yet instructive process.

Verification Process: Step-by-Step Approach

The core method for verification involves substitution: plugging each candidate function into the original differential equation and checking whether the equation holds true. This process typically follows these steps:

1. Compute the derivatives: Find the first and second derivatives of the function.
2. Substitute into the differential equation: Plug the derivatives and the original function into the equation.
3. Simplify and compare: Check whether the resulting expression simplifies to zero, confirming the function as a solution.

Let's analyze each function individually.

Verifying $Y_1(t) = e^t$

Step 1: Compute derivatives

- First derivative:

$$ Y_1'(t) = \frac{d}{dt} e^t = e^t $$

- Second derivative:

$$ Y_1''(t) = \frac{d}{dt} e^t = e^t $$

Step 2: Substitute into the differential equation

Plugging into $Y'' - Y = 0$:

$$ Y_1''(t) - Y_1(t) = e^t - e^t = 0 $$

Step 3: Simplify and verify

The expression simplifies precisely to zero, confirming that $Y_1(t) = e^t$ satisfies the differential equation.

Verifying $Y_2(t) = e^{-t}$

Step 1: Compute derivatives

- First derivative:

$$ Y_2'(t) = \frac{d}{dt} e^{-t} = -e^{-t} $$

- Second derivative:

$$ Y_2''(t) = \frac{d}{dt} (-e^{-t}) = e^{-t} $$

Step 2: Substitute into the differential equation

Plugging into $Y'' - Y = 0$:

$$ Y_2''(t) - Y_2(t) = e^{-t} - e^{-t} = 0 $$

Step 3: Simplify and verify

Again, the expression simplifies to zero, confirming $Y_2(t) = e^{-t}$ as a solution.

Consolidated Findings and Significance

The explicit verification confirms that both functions:

- $Y_1(t) = e^t$
- $Y_2(t) = e^{-t}$

are solutions of the differential equation $Y'' - Y = 0$. Moreover, these functions form a fundamental set of solutions, implying that the general solution can be expressed as:

$$ Y(t) = C_1 e^t + C_2 e^{-t} $$

where C_1 and C_2 are arbitrary constants, allowing for modeling a broad spectrum of initial conditions.

Implications for Differential Equation Solution Strategies

This process of verification underscores several critical points relevant to the study and application of differential equations:

1. Matching Candidate Solutions to Characteristic Roots
The solutions e^t and e^{-t} directly correspond to the roots of the characteristic equation. Recognizing this pattern expedites the solution process and simplifies verification.

2. Use of Derivatives for Verification
Calculating derivatives and substituting back into the original equation is a straightforward, reliable method to verify solutions, especially for exponential functions where derivatives are proportional to the original function.

3. Confirming Completeness of Solution Sets
Since the solutions satisfy the differential equation, they are part of the fundamental solution set. This is vital for constructing general solutions and understanding the system's behavior.

4. Application in Modeling and Engineering
Verifying solutions ensures accuracy in modeling physical systems such as mechanical vibrations, electrical circuits, and thermal processes, where differential equations describe system dynamics.

Additional Considerations and Advanced Topics

While the verification process for exponential solutions like e^t and e^{-t} is straightforward, more complex differential equations may involve functions such as sines, cosines, or special functions. In such cases, verification might involve:

- Using identities and properties of functions (e.g., trigonometric identities)
- Applying variation of parameters or undetermined coefficients for nonhomogeneous equations
- Utilizing numerical methods when analytical solutions are intractable

Furthermore, understanding the structure of the differential equation helps in identifying solutions and their linear independence, critical for forming the general solution.

Conclusion

The process of verifying solutions to differential equations is a fundamental skill that ensures mathematical rigor and accuracy in modeling real-world phenomena. The case of the differential equation $Y'' - Y = 0$ and the functions e^t and e^{-t} exemplifies how direct substitution and differentiation confirm solution validity. Recognizing these functions as solutions not only validates their correctness but also provides insight into the structure of the general solution. As such, mastering the verification process enhances one's ability to analyze, interpret, and utilize differential equations across various scientific and engineering disciplines.

In essence, verifying solutions is akin to a product review for mathematical functions—ensuring they meet the specifications set by the differential equation and are fit for application in complex models.

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