Verify That The Divergence Theorem Is True For The Vector Field F On The Region E. Give The Flux. F(x,y,z)

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The Divergence Theorem, also known as Gauss's Theorem, is a fundamental result in vector calculus that relates the flux of a vector field across a closed surface to the divergence of the field within the volume enclosed by that surface. Verifying this theorem for a specific vector field over a given region not only consolidates understanding of its application but also demonstrates its utility in evaluating flux integrals more efficiently. In this article, we will explore the steps necessary to verify the Divergence Theorem for a particular vector field $\mathbf{F}$ over a region E, compute the flux $\iint_{S} \mathbf{F} \cdot \mathbf{n} \, dS$, and understand the underlying concepts involved.

Understanding the Divergence Theorem

The Divergence Theorem states that for a continuously differentiable vector field $\mathbf{F}$ defined on a region E with boundary surface S, the following equality holds:

$$
\iiint_{E} (\nabla \cdot \mathbf{F}) \, dV = \iint_{S} \mathbf{F} \cdot \mathbf{n} \, dS
$$

where:
- $\nabla \cdot \mathbf{F}$ is the divergence of $\mathbf{F}$,
- dV is the volume element,
- $\mathbf{n}$ is the outward-pointing unit normal vector on the surface S,
- dS is the surface element.

The theorem essentially states that the total "outflow" of the vector field across the boundary surface S equals the total divergence within the volume E.

Choosing the Vector Field $\mathbf{F}$ and Region E

To verify the theorem, we need to specify:
- a vector field $\mathbf{F}$,
- a region E with a well-defined boundary surface S.

Example Vector Field:

Let’s consider a simple and illustrative vector field:

$$
\mathbf{F}(x, y, z) = (x^2, y^2, z^2)
$$

Example Region E:

Choose the region to be the unit cube:

$$
E = \{(x, y, z) \mid 0 \leq x \leq 1,\, 0 \leq y \leq 1,\, 0 \leq z \leq 1 \}
$$

The boundary surface S of this cube consists of six faces, each with a well-defined outward normal vector.

Calculating the Divergence $\nabla \cdot \mathbf{F}$

The divergence of $\mathbf{F} = (x^2, y^2, z^2)$ is:

$$
\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) = 2x + 2y + 2z
$$

This scalar function represents the "source strength" at each point within E.

Computing the Volume Integral $\iiint_{E} (\nabla \cdot \mathbf{F}) \, dV$

Since E is the unit cube with bounds from 0 to 1 in each coordinate:

$$
\iiint_{E} (2x + 2y + 2z) \, dV = \int_0^1 \int_0^1 \int_0^1 (2x + 2y + 2z) \, dx\, dy\, dz
$$

This integral can be separated:

$$
= 2 \left(\int_0^1 x \, dx \int_0^1 dy \int_0^1 dz \right) + 2 \left(\int_0^1 y \, dy \int_0^1 dx \int_0^1 dz \right) + 2 \left(\int_0^1 z \, dz \int_0^1 dx \int_0^1 dy \right)
$$

Computing each:

$$
\int_0^1 x \, dx = \frac{1}{2}, \quad \int_0^1 y \, dy = \frac{1}{2}, \quad \int_0^1 z \, dz = \frac{1}{2}
$$

and

$$
\int_0^1 dx = \int_0^1 dy = \int_0^1 dz = 1
$$

Thus, the volume integral becomes:

$$
= 2 \times \frac{1}{2} \times 1 \times 1 + 2 \times \frac{1}{2} \times 1 \times 1 + 2 \times \frac{1}{2} \times 1 \times 1 = 1 + 1 + 1 = 3
$$

Result:

$$
\iiint_{E} (\nabla \cdot \mathbf{F}) \, dV = 3
$$

This is the total "source strength" within the cube.

Calculating the Surface Flux $\iint_{S} \mathbf{F} \cdot \mathbf{n} \, dS$

Next, we evaluate the flux across the boundary surface S. Since S consists of six faces, we compute the flux over each face and sum the results.

Normal Vectors and Faces:

1. Face at $x=0$: outward normal $\mathbf{n} = (-1, 0, 0)$
2. Face at $x=1$: outward normal $\mathbf{n} = (1, 0, 0)$
3. Face at $y=0$: outward normal $\mathbf{n} = (0, -1, 0)$
4. Face at $y=1$: outward normal $\mathbf{n} = (0, 1, 0)$
5. Face at $z=0$: outward normal $\mathbf{n} = (0, 0, -1)$
6. Face at $z=1$: outward normal $\mathbf{n} = (0, 0, 1)$

Flux over each face:

1. Face at $x=1$:

$$
\mathbf{F} = (1^2, y^2, z^2) = (1, y^2, z^2)
$$
$$
\mathbf{F} \cdot \mathbf{n} = (1, y^2, z^2) \cdot (1, 0, 0) = 1
$$

Integral over $y, z \in [0, 1]$:

$$
\iint_{0}^{1} \iint_{0}^{1} 1 \, dy\, dz = 1 \times 1 = 1
$$

---

2. Face at $x=0$:

$$
\mathbf{F} = (0, y^2, z^2)
$$
$$
\mathbf{F} \cdot \mathbf{n} = (0, y^2, z^2) \cdot (-1, 0, 0) = 0
$$
$$
\Rightarrow \text{Flux} = 0
$$

---

3. Face at $y=1$:

$$
\mathbf{F} = (x^2, 1, z^2)
$$
$$
\mathbf{F} \cdot \mathbf{n} = (x^2, 1, z^2) \cdot (0, 1, 0) = 1
$$
Integral over $x, z \in [0, 1]$:

$$
\iint_{0}^{1} \iint_{0}^{1} 1 \, dx\, dz = 1
$$

---

4. Face at $y=0$:

$$
\mathbf{F} = (x^2, 0, z^2)
$$
$$
\mathbf{F} \cdot \mathbf{n} = (x^2, 0, z^2) \cdot (0, -1, 0) = 0
$$
Flux = 0.

---

5. Face at $z=1$:

$$
\mathbf{F} = (x^2, y^2, 1)
$$
$$
\mathbf{F} \cdot \mathbf{n} = (x^2, y^2, 1) \cdot (0, 0, 1)
$$

Frequently Asked Questions

What is the Divergence Theorem and how does it apply to vector fields like F(x, y, z)?

The Divergence Theorem states that the flux of a vector field F across a closed surface S is equal to the triple integral of the divergence of F over the volume E enclosed by S. Mathematically, ∭_E (div F) dV = ∬_S F · n dS. It applies to vector fields like F(x, y, z) to relate surface integrals to volume integrals, simplifying flux calculations.

How do you verify that the Divergence Theorem holds for a specific vector field F on region E?

To verify the theorem, compute the divergence of F, integrate it over the volume E to find the volume integral, and then compute the flux of F across the boundary surface S. If both results are equal, the Divergence Theorem holds for F on E.

What are the steps to compute the flux of F(x, y, z) across a closed region E?

First, identify the boundary surface S of E. Then, compute the surface integral ∬_S F · n dS directly or verify via the Divergence Theorem by calculating the volume integral of div F over E. The flux is the surface integral of F · n over S.

Can you give an example of a vector field F(x, y, z) and a region E where you verify the Divergence Theorem?

Yes. For example, F(x, y, z) = (x, y, z) and E as the unit sphere $x^2 + y^2 + z^2 \leq 1$. Compute div F = 3, integrate over E to get the volume integral, and compute the flux directly over the surface. Both results should match, confirming the theorem.

What is the significance of the divergence in verifying the Divergence Theorem for F(x, y, z)?

The divergence of F measures the net outflow of the field at each point. By integrating divergence over E, we quantify total outflow, which the Divergence Theorem equates to the flux across the boundary surface, making divergence central to the theorem's verification.

How do boundary conditions of region E affect the verification of the Divergence Theorem for F?

The boundary conditions define the shape and surface S of E. Accurate parameterization of S and ensuring F is well-behaved (continuous, differentiable) on E and its boundary are essential for valid application and verification of the Divergence Theorem.

What are common pitfalls when verifying the

Divergence Theorem for a vector field F?

Common pitfalls include incorrect calculation of divergence, improper parameterization of the surface S, neglecting surface orientation, or misapplying the divergence theorem to non-closed regions. Ensuring correct calculations and boundary conditions is crucial.

Additional Resources

Divergence Theorem Verification: An Expert Examination of the Flux of \(\mathbf{F}(x, y, z)\) on Region \(E\)

Introduction: Understanding the Divergence Theorem

The Divergence Theorem, also known as Gauss's Theorem or Gauss's Divergence Theorem, is a fundamental result in vector calculus. It provides a powerful link between the flux of a vector field across a closed surface and the divergence of the field within the volume enclosed by that surface. This theorem is instrumental in physics, engineering, and mathematics, especially in fields dealing with fluid flow, electromagnetism, and continuum mechanics.

Before delving into the verification process, let's clarify the core concepts:

- Flux: The total "flow" of a vector field through a surface.
- Divergence: A scalar measure of the "source" or "sink" strength at a point within a field.
- Region \(E\): The volume enclosed by a closed surface in three-dimensional space.
- Vector Field \(\mathbf{F}(x, y, z)\): A function assigning a vector to each point in space.

The goal of this article is to verify that the Divergence Theorem holds for a specific vector field \(\mathbf{F}\) over a region \(E\), and to compute the flux explicitly to demonstrate this.

Defining the Vector Field \(\mathbf{F}\) and Region \(E\)

The Vector Field \(\mathbf{F}(x, y, z)\)

For our analysis, consider a vector field of the form:

\[

$$
\mathbf{F}(x, y, z) = P(x, y, z) \, \mathbf{i} + Q(x, y, z) \, \mathbf{j} + R(x, y, z) \, \mathbf{k}
$$

where:

$$
P(x, y, z) = x^2, \quad Q(x, y, z) = y^2, \quad R(x, y, z) = z^2
$$

This choice is deliberate for its simplicity and illustrative power. The quadratic dependence ensures non-trivial divergence while maintaining manageable calculations.

The Region E

Let E be the solid unit cube:

$$
E = \{ (x, y, z) \mid 0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad 0 \leq z \leq 1 \}
$$

This compact, bounded region is ideal for our verification because:

- It is a standard, well-understood shape.
- Its boundary surface is composed of six flat faces.
- The limits of integration are straightforward.

Step 1: Computing the Divergence $\nabla \cdot \mathbf{F}$

The divergence of $\mathbf{F}$ is given by:

$$
\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}
$$

Calculating each term:

- $\frac{\partial P}{\partial x} = \frac{\partial x^2}{\partial x} = 2x$
- $\frac{\partial Q}{\partial y} = \frac{\partial y^2}{\partial y} = 2y$
- $\frac{\partial R}{\partial z} = \frac{\partial z^2}{\partial z} = 2z$

Thus,

$$
\nabla \cdot \mathbf{F} = 2x + 2y + 2z
$$

This divergence quantifies how much the vector field $\mathbf{F}$ "sources" or "sinks" within the region E.

Step 2: Computing the Volume Integral $\iiint_E (\nabla \cdot \mathbf{F}) \, dV$

According to the Divergence Theorem:

$$
\iint_{\partial E} \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E \nabla \cdot \mathbf{F} \, dV
$$

where ∂E is the boundary surface of E, and $\mathbf{n}$ is the outward unit normal vector.

Let's compute the volume integral of the divergence:

$$
\iiint_E (2x + 2y + 2z) \, dV
$$

Given the symmetry and the limits:

$$
x, y, z \in [0, 1]
$$

the integral separates into three parts:

$$
2 \iiint_E x \, dV + 2 \iiint_E y \, dV + 2 \iiint_E z \, dV
$$

Calculate each:

$$
\iiint_E x \, dV = \int_0^1 \int_0^1 \int_0^1 x \, dz \, dy \, dx
$$

Since the integrand does not depend on z or y:

$$
= \left(\int_0^1 dz \right) \left(\int_0^1 dy \right) \left(\int_0^1 x \, dx \right) = (1)(1) \times \left[\frac{x^2}{2} \Big|_0^1 \right] = \frac{1}{2}
$$

Similarly,

$$
\iiint_E y \, dV = \frac{1}{2}
$$

and

$$
\iiint_E z \, dV = \frac{1}{2}
$$

Putting it all together:

$$
\iiint_E (\nabla \cdot \mathbf{F}) \, dV = 2 \times \frac{1}{2} + 2 \times \frac{1}{2} + 2 \times \frac{1}{2} = 1 + 1 + 1 = 3
$$

Result: The volume integral of the divergence over E is 3.

Step 3: Computing the Surface Integral (Flux) $\iint_{\partial E} \mathbf{F} \cdot \mathbf{n} \, dS$

To verify the theorem, we now compute the flux directly over the boundary surface ∂E. The boundary consists of six faces:

- Face at $x=0$, $x=1$
- Face at $y=0$, $y=1$
- Face at $z=0$, $z=1$

We will evaluate the flux across each face and sum the results.

Flux Across the Faces at $x=0$ and $x=1$

- At $x=0$:

$$
\mathbf{F} = (0^2, y^2, z^2) = (0, y^2, z^2)
$$

- Normal vector: $\mathbf{n} = -\mathbf{i}$ (pointing outward, i.e., in the negative x-direction).

- Surface element: $dS = dy\,dz$

Flux:

$$
\Phi_{x=0} = \iint_{y=0}^{1} \int_{z=0}^{1} \mathbf{F} \cdot \mathbf{n} \, dy\,dz = \iint_{y=0}^1 \int_{z=0}^1 (0, y^2, z^2) \cdot (-1, 0, 0) \, dy\,dz = 0
$$

since the dot product involves only the x-component.

- At $x=1$:

\[
\mathbf{F} = (1^2, y^2, z^2) = (1, y^2, z^2)
\]

- Normal vector: \(\mathbf{n} = \mathbf{i}\) (outward in positive \(x\)-direction).

Flux:

\[
\Phi_{x=1} = \iint_{0}^{1} \int_{0}^{1} (1, y^2, z^2) \cdot (1, 0, 0) \, dy\,dz = \iint_{0}^{1} \int_{0}^{1} 1 \, dy\,dz = 1 \times 1 = 1
\]

Flux Across the Faces at \(y=0\) and \(y=1\)

- At \(y=0\):

\[
\mathbf{F} = (x^2, 0, z^2)
\]

- Normal vector: \(\mathbf{n} = -\mathbf{j}\).

Flux:

\[
\Phi_{y=0} = \iint_{0}^{1} \int_{0}^{

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