Consider The Set S = {(x, Y, Z) | 0 X 1, 0 Y 2x, 0 Z X + 3y}. Prove That S Is A Jordan Region And Integrate

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Understanding the nature of specific regions in space and their properties is fundamental in multivariable calculus. In this article, we will explore the set $S = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq 2x, 0 \leq z \leq x + 3y\}$, analyze whether it qualifies as a Jordan region, and calculate the corresponding triple integral over this domain. By systematically examining each aspect, we will demonstrate that S is indeed a Jordan region and compute integrals over it, providing insights into the geometric and analytical properties involved.

Understanding the Set S

Definition of the Set S

The set S is defined by three inequalities involving variables x, y, z:

- $0 \leq x \leq 1$
- $0 \leq y \leq 2x$
- $0 \leq z \leq x + 3y$

These inequalities specify a three-dimensional region in space, bounded by planes and surfaces determined by the equations.

Geometric Interpretation

- The variable x ranges from 0 to 1, forming a vertical slab along the x-axis.
- For each fixed x, y varies from 0 up to $2x$: this creates a triangular cross-section in the xy-plane, with vertices at $(0,0)$ and $(x, 2x)$.
- For each fixed (x, y), z varies from 0 up to $x + 3y$, which is a plane slanting in the z-direction.

Visualizing this region involves understanding how these bounds shape the volume in three-dimensional space.

Proving S Is a Jordan Region

What Is a Jordan Region?

A Jordan region in $\mathbb{R}^3$ is a bounded, regular, and measurable region with boundary that is a finite union of Lipschitz surfaces (or equivalently, piecewise smooth surfaces). In simpler terms, it is a solid volume with a well-defined boundary that can be approximated by finite unions of smooth surfaces.

Step-by-Step Proof that S Is a Jordan Region

To establish S as a Jordan region, we need to verify:

1. Boundedness: S is contained within a finite region.
2. Closure and Interior: S is closed and contains its interior points.
3. Boundary Regularity: The boundary is composed of finitely many smooth surfaces.

Let's examine these points:

1. Boundedness

- Since x ranges from 0 to 1, the domain is confined along the x-axis.
- y is bounded between 0 and $2x$, which, given $x \leq 1$, implies $y \leq 2$.
- z varies between 0 and $x + 3y$, which is bounded above by $1 + 3 \times 2 = 1 + 6 = 7$.
- Therefore, S is contained within the cube $[0, 1] \times [0, 2] \times [0, 7]$. Hence, S is bounded.

2. Closure and Interior

- The inequalities are non-strict ($\leq$), so S includes its boundary points.
- The interior points are those with strict inequalities, and the boundary points satisfy equalities.
- Since the defining inequalities are continuous and involve smooth surfaces (planes), the boundary is well-defined and closed.
- Therefore, S is a closed, bounded set with a non-empty interior.

3. Boundary Regularity

- The boundary surfaces are given by:

- $x = 0$ and $x = 1$,
- $y = 0$ and $y = 2x$,

- $z = 0$ and $z = x + 3y$.

- These are planes (or lines in the case of intersections), which are smooth surfaces.
- The intersection of these surfaces forms a finite union of polygons, which are Lipschitz surfaces.

Since all these conditions are satisfied, S qualifies as a Jordan region.

Describing the Region S in Terms of Integration

Order of Integration and Limits

To evaluate integrals over S, it is essential to set up the proper limits in an order that simplifies calculations. Typically, choosing the order $dz\, dy\, dx$ or $dy\, dz\, dx$ depends on the bounds.

Given the inequalities:

- x varies from 0 to 1,
- y varies from 0 to $2x$,
- z varies from 0 to $x + 3y$.

The natural order is:

$$x \in [0, 1], \quad y \in [0, 2x], \quad z \in [0, x + 3y]$$

Thus, the triple integral over S can be written as:

$$
\iiint_S f(x, y, z) \, dV = \int_{x=0}^1 \int_{y=0}^{2x} \int_{z=0}^{x + 3y} f(x, y, z) \, dz \, dy \, dx
$$

This setup simplifies the computation of integrals over the region.

Calculating the Volume of S

The volume of the region S is obtained by integrating the constant function $f(x, y, z) = 1$:

$$
V = \iiint_S 1 \, dV = \int_{x=0}^1 \int_{y=0}^{2x} \int_{z=0}^{x + 3y} 1 \, dz \, dy \, dx
$$

Let's compute this step-by-step:

1. Integrate with respect to z:

$$
\int_{z=0}^{x + 3y} 1 \, dz = x + 3y
$$

2. Next, integrate with respect to y:

$$
\int_{0}^{2x} (x + 3y) \, dy
$$

Calculate:

$$
\int_{0}^{2x} x \, dy + \int_{0}^{2x} 3y \, dy = x \times (2x - 0) + 3 \times \frac{(2x)^2}{2} = 2x^2 + 3 \times \frac{4x^2}{2} = 2x^2 + 3 \times 2x^2 = 2x^2 + 6x^2 = 8x^2
$$

3. Finally, integrate with respect to x:

$$
\int_{0}^{1} 8x^2 \, dx = 8 \times \frac{x^3}{3} \bigg|_0^1 = 8 \times \frac{1}{3} = \frac{8}{3}
$$

Result:

$$
\boxed{
\text{Volume of } S = \frac{8}{3}
}
$$

This confirms that S is a finite, bounded volume with a well-defined measure.

Integrating a Function over S

Suppose we want to evaluate the integral of a continuous function $f(x, y, z)$ over S. The setup is:

$$
\iiint_S f(x, y, z) \, dV = \int_{x=0}^1 \int_{y=0}^{2x} \int_{z=0}^{x + 3y} f(x, y, z) \, dz \, dy \, dx
$$

Let's consider a specific example: integrating $f(x, y, z) = z$.

Step-by-step calculation:

1. Integrate z over $z \in [0, x + 3y]$:

$$
\int_{0}^{x + 3y} z \, dz = \frac{(x + 3y)^2}{2}
$$

2. Integrate with respect to y:

$$
\int_{0}^{2x} \frac{(x + 3y)^2}{2} \, dy
$$

Expand numerator:

\[
(x

Frequently Asked Questions

What is the set S defined as in the problem?

The set S is defined as $S = \{(x, y, z) \mid 0 \le x \le 1, 0 \le y \le 2x, 0 \le z \le x + 3y\}$.

How do we verify that S is a Jordan region?

To verify that S is a Jordan region, we need to show that S is bounded and its boundary is a continuous, piecewise smooth surface, which can be done by analyzing the inequalities and their boundaries.

What are the boundary surfaces of set S?

The boundary surfaces are: $x = 0$, $x = 1$, $y = 0$, $y = 2x$, $z = 0$, and $z = x + 3y$, corresponding to the limits of the inequalities.

How can we describe the projection of S onto the xy-plane?

The projection onto the xy-plane is the region $R = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 2x\}$.

What is the method to compute the volume of set S?

The volume can be computed using a triple integral over S, typically integrating z from 0 to $x + 3y$, y from 0 to $2x$, and x from 0 to 1.

How do we set up the integral to find the volume of S?

The volume $V = \int_0^1 \int_0^{2x} \int_0^{x+3y} dz\, dy\, dx$, which simplifies to $V = \int_0^1 \int_0^{2x} (x + 3y)\, dy\, dx$.

What is the value of the volume after evaluating the integral?

Evaluating the integral yields V = 1.75 cubic units, confirming the volume of the region S.

Additional Resources

Consider The Set S = {(x, y, z) | 0 ≤ x ≤ 1, 0 ≤ y ≤ 2x, 0 ≤ z ≤ x + 3y}: A Deep Dive into Geometric Properties, Jordan Regions, and Multiple Integrals

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Introduction

The study of geometric regions in multivariable calculus is fundamental not only for understanding the spatial structures involved but also for applying integration techniques effectively. The set $S = \{ (x, y, z) \mid 0 \leq x \leq 1, \ 0 \leq y \leq 2x, \ 0 \leq z \leq x + 3y \}$ presents an intriguing example of a three-dimensional region bounded by inequalities that depend on each other.

This article aims to rigorously analyze this set, demonstrate that it constitutes a Jordan region, and perform the relevant integrations over S. The investigation proceeds through a detailed examination of the set’s boundaries, the properties that qualify it as a Jordan region, and the computation of integrals over it, emphasizing clarity and mathematical precision.

Understanding the Set S: Geometric and Analytical Perspectives

The Definition of S

The set is defined by the inequalities:

$$
\begin{cases}
0 \leq x \leq 1, \\
0 \leq y \leq 2x, \\
0 \leq z \leq x + 3y.
\end{cases}
$$

This describes a region in $\mathbb{R}^3$ where the variables are interdependent, with the bounds on y and z expressed as functions of x and y, respectively.

Visualizing the Region

Step 1: The xy-projection

- The inequalities $0 \leq x \leq 1$ and $0 \leq y \leq 2x$ define a region in the xy-plane.

- The boundary in the xy-plane is given by the lines:

- $y=0$ (the x-axis),
- $y=2x$ (a line passing through the origin with slope 2),
- $x=1$ (a vertical boundary).

- The region in the xy-plane is a right triangle bounded by:

- $x=0$, $y=0$,
- $y=2x$ (from $x=0$ to $x=1$),
- $x=1$.

Step 2: The z-bounds

- For each fixed point (x, y) in this region, z ranges from 0 up to $x + 3 y$.

- The upper surface $z = x + 3 y$ is a plane in $\mathbb{R}^3$, slanting upward depending on x and y.

Formal Proof That S Is a Jordan Region

What Is a Jordan Region?

A subset $R \subset \mathbb{R}^n$ is called a Jordan measurable region (or Jordan region) if:

1. It is bounded.
2. Its boundary is a finite union of finitely many surfaces with continuous (or at least piecewise continuous) parametrizations.
3. Its boundary has measure zero (i.e., the boundary does not contribute to volume).

In simpler terms, Jordan regions are "well-behaved" regions for integration, with "nice" boundaries.

Step 1: Boundedness

- The inequalities restrict x to $[0,1]$, y to $[0, 2x] \subset [0, 2]$, and z to $[0, x + 3 y]$.

- Since $x \leq 1$, $y \leq 2$, and $z \leq x + 3 y \leq 1 + 3 \times 2 = 7$, the entire set S is contained within a cube $[0,1] \times [0,2] \times [0,7]$.

- Therefore, S is bounded.

Step 2: Boundary Regularity

The boundary of S consists of:

- The "bottom" surface $z=0$,
- The "top" surface $z = x + 3 y$,
- The "lateral" boundaries at $x=0$, $y=2x$, and $x=1$.

Each of these surfaces is:

- Defined by continuous, smooth functions,
- Piecewise smooth (planes and linear inequalities).

Thus, the boundary is a union of finitely many surfaces with continuous parametrizations.

Step 3: Closure and Interior

Since the inequalities are closed (inclusive bounds), the set S is closed. Its interior is non-empty, and the boundary has Lebesgue measure zero in $\mathbb{R}^3$, satisfying the Jordan criteria.

Conclusion:

The set S is bounded, closed, and its boundary consists of finitely many smooth (planar) surfaces. Therefore, S is a Jordan measurable region in $\mathbb{R}^3$.

Parameterization and Integration over S

Having established that S is a Jordan region, the next step involves integrating functions over S. This requires setting up the integrals in a suitable order, based on the inequalities.

Integration Strategy

Given the bounds:

$$
0 \leq x \leq 1,
$$
$$
0 \leq y \leq 2x,
$$
$$
0 \leq z \leq x + 3 y,

\]

a natural choice is to integrate in the order \(dz \, dy \, dx \).

Computing the Volume of \(S \)

The volume \(V \) of the region is given by:

\[
V = \iiint_{S} dV.
\]

Expressed as an iterated integral:

\[
V = \int_{x=0}^{1} \int_{y=0}^{2x} \int_{z=0}^{x + 3 y} dz \, dy \, dx.
\]

Step 1: Integrate with respect to \(z \):

\[
\int_{0}^{x + 3 y} dz = x + 3 y.
\]

Step 2: Integrate with respect to \(y \):

\[
\int_{0}^{2x} (x + 3 y) dy = \int_{0}^{2x} x \, dy + \int_{0}^{2x} 3 y \, dy.
\]

Calculations:

\[
x \times (2x - 0) = 2 x^2,
\]

\[
3 \times \frac{(2x)^2}{2} = 3 \times \frac{4 x^2}{2} = 3 \times 2 x^2 = 6 x^2.
\]

Adding:

\[
2 x^2 + 6 x^2 = 8 x^2.
\]

Step 3: Integrate with respect to \(x \):

\[

V = \int_{0}^{1} 8 x^2 \, dx = 8 \times \frac{x^3}{3} \Big|_{0}^{1} = 8 \times \frac{1}{3} = \frac{8}{3}.
\]

Therefore, the volume of \(S \) is \(\boxed{\frac{8}{3}} \).

Integration of Functions over \(S \): An Example

Suppose we are asked to evaluate the integral of the function \(f(x, y, z) = z \) over \(S \):

\[
I = \iiint_{S} z \, dV.
\]

Setup:

\[
I = \int_{x=0}^{1} \int_{y=0}^{2x} \int_{z=0}^{x + 3 y} z \, dz \, dy \, dx.
\]

Step 1: Integrate with respect to \(z \):

\[
\int_{0}^{x + 3 y} z \, dz = \frac{(x + 3 y)^2}{2}.
\]

Step 2: Integrate with respect to \(y \):

\[
\int_{0}^{2x} \frac{(x + 3 y)^2}{2} dy = \frac{1}{2} \int_{0}^{2x} (x + 3 y)^2 dy.
\]

Expanding \((x + 3 y)^2 \):

\[
x^2 + 6 x y + 9 y^2.
\]

Integrate term-by-term:

\[
\int_{0}^{2x} x^2 dy = x^2 \times 2x = 2 x^3,
\]

\[
\int_{0}^{2x} 6

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