Find A Polynomial P3 Such That {Po. PI. P2. P3) (see Ex- Ercise 11) Is An Orthogonal Basis For The Subspace

Find A Polynomial P3 Such That {Po, P1, P2, P3} (see Exercise 11) Is An Orthogonal Basis For The Subspace

In the study of polynomial spaces, constructing an orthogonal basis is a fundamental task that simplifies many problems in approximation theory, numerical analysis, and functional analysis. When working within a subspace of polynomials, such as those of degree less than or equal to a certain degree, developing an orthogonal basis allows for easier computation of projections, least squares approximations, and understanding of the subspace's structure. This article will guide you through the process of finding a polynomial P_3 such that the set $\{ P_0, P_1, P_2, P_3 \}$ forms an orthogonal basis for a specific subspace, building upon the initial polynomials P_0, P_1, P_2.

Understanding the Context and Notation

The Polynomial Space and Its Subspace

Typically, the polynomials P_0, P_1, P_2, P_3 are elements of a polynomial space, such as P_n, which contains all polynomials up to degree n. For the purposes of this discussion, we assume:

- $P_0(t)$ is a polynomial of degree 0 (a constant).
- $P_1(t)$ is a polynomial of degree 1.
- $P_2(t)$ is a polynomial of degree 2.
- $P_3(t)$ will be a polynomial of degree 3, to extend the basis.

The goal is to choose $P_3(t)$ such that the entire set becomes orthogonal under a given inner product, often defined as:

$$
\langle f, g \rangle = \int_a^b f(t) g(t) w(t) dt
$$

where $w(t)$ is a weight function, and $[a, b]$ is an interval of integration.

Exercise 11 and Its Significance

Exercise 11, in typical contexts, might involve given polynomials P_0, P_1, P_2, and the task is to orthogonalize them or extend the basis. Usually, this involves the Gram-Schmidt process, which systematically orthogonalizes a set of functions (or polynomials) with respect to the inner product.

The Gram-Schmidt Process for Polynomials

Overview of the Process

The Gram-Schmidt process is a method to convert a set of linearly independent vectors (or functions) into an orthogonal set. When applied to polynomials, the process involves:

1. Starting with the initial basis polynomials.
2. Orthogonalizing each polynomial against all previous ones.
3. Normalizing if an orthonormal basis is desired.

For our purpose, normalization is not necessary; orthogonality suffices.

Applying Gram-Schmidt to P_0, P_1, P_2

Suppose we have:

- $P_0(t)$
- $P_1(t)$
- $P_2(t)$

We want to find $P_3(t)$ so that the set

$$
\{ P_0, P_1, P_2, P_3 \}
$$

is orthogonal under the inner product.

Step-by-Step Construction of P_3

Step 1: Starting with $P_0(t)$

Assuming $P_0(t) = 1$, a typical starting point, as it is a constant polynomial.

Step 2: Orthogonalize $P_1(t)$

Suppose $P_1(t)$ is linear, like t. To orthogonalize:

$$
Q_1(t) = P_1(t) - \frac{\langle P_1, P_0 \rangle}{\langle P_0, P_0 \rangle} P_0(t)
$$

which ensures $Q_1(t)$ is orthogonal to $P_0(t)$.

Step 3: Orthogonalize $P_2(t)$

Similarly,

$$
Q_2(t) = P_2(t) - \frac{\langle P_2, P_0 \rangle}{\langle P_0, P_0 \rangle} P_0(t) - \frac{\langle P_2, Q_1 \rangle}{\langle Q_1, Q_1 \rangle} Q_1(t)
$$

so $Q_2(t)$ is orthogonal to both $P_0(t)$ and $Q_1(t)$.

Step 4: Construct $P_3(t)$

Finally, to find $P_3(t)$, choose a polynomial of degree 3, say $t^3

\), and orthogonalize it against all previous P_i:

$$
P_3(t) = P_3^{\text{raw}}(t) - \sum_{k=0}^2 \frac{\langle P_3^{\text{raw}}, P_k \rangle}{\langle P_k, P_k \rangle} P_k(t)
$$

where $P_3^{\text{raw}}(t)$ could be t^3, and the subtraction ensures orthogonality.

Choosing $P_3(t)$: Practical Example

Common Choices for P_0, P_1, P_2

For simplicity, assume the standard inner product over $[-1, 1]$ with weight $w(t) = 1$:

$$
\langle f, g \rangle = \int_{-1}^1 f(t) g(t) dt
$$

In this setting, the classical orthogonal polynomials are Legendre polynomials:

- $P_0(t) = 1$
- $P_1(t) = t$
- $P_2(t) = \frac{1}{2}(3t^2 - 1)$

These are orthogonal with respect to the standard inner product over $[-1,1]$. Therefore, the next polynomial $P_3(t)$ in the Legendre sequence is:

$$
P_3(t) = \frac{1}{2}(5t^3 - 3t)
$$

which is orthogonal to the previous ones.

Verifying Orthogonality

To verify, compute inner products such as:

$$
\langle P_3, P_0 \rangle, \quad \langle P_3, P_1 \rangle, \quad \langle P_3, P_2 \rangle
$$

and confirm they are zero.

General Approach for Finding P_3

1. Define the Inner Product

Select the appropriate inner product based on the problem's context, often an integral over a specific interval with a weight function.

2. Use Known Orthogonal Polynomials

If the initial polynomials P_0, P_1, P_2 are not standard, express them explicitly, then apply Gram-Schmidt orthogonalization to derive P_3.

3. Orthogonalize the Candidate Polynomial

Choose a polynomial of degree 3, such as t^3, and orthogonalize it against the existing basis:

$$
P_3(t) = t^3 - \sum_{k=0}^2 \frac{\langle t^3, P_k \rangle}{\langle P_k, P_k \rangle} P_k(t)
$$

This ensures the set is orthogonal.

4. Normalize if Required

To obtain an orthonormal basis, normalize $P_3(t)$ by dividing by its norm:

$$
\| P_3 \| = \sqrt{\langle P_3, P_3 \rangle}
$$

5. Confirm Orthogonality

Verify that:

$$
\langle P_3, P_k \rangle = 0 \quad \text{for} \quad k=0,1,2
$$

Applications and Significance

Constructing orthogonal polynomial bases like the Legendre, Chebyshev, or Laguerre polynomials is critical in various applications:

- Numerical Integration: Gaussian quadrature relies on orthogonal polynomials to select nodes and weights.
- Approximation Theory: Polynomial approximations of functions become more efficient and accurate using orthogonal bases.
- Spectral Methods: Solving differential equations numerically often involves expanding solutions in terms of orthogonal polynomials.

Having a systematic procedure to find P_3, or higher-degree polynomials, ensures that these powerful tools can be extended to custom function spaces or problem-specific weight functions.

Conclusion

Finding a polynomial P_3 such that $\{ P_0, P_1, P_2, P_3 \}$ forms an orthogonal basis involves understanding the underlying inner product space, applying the Gram-Schmidt process, and selecting appropriate initial polynomials. When dealing with classical polynomial families like Legendre, Chebyshev, or

Frequently Asked Questions

What is the main goal when finding a polynomial P3 to extend an orthogonal basis {P0, P1, P2}?

The main goal is to find a polynomial P3 such that the set {P0, P1, P2, P3} remains orthogonal with respect to the given inner product, thereby forming an orthogonal basis for the subspace.

How do we verify that the set {P0, P1, P2, P3} is an orthogonal basis?

We verify orthogonality by checking that the inner product of P3 with each of P0, P1, and P2 is zero, and ensure that all the polynomials are linearly independent.

What is the typical method to construct P3 given the existing orthogonal polynomials?

Typically, P3 is constructed using the Gram-Schmidt process or by orthogonalizing a polynomial of higher degree against the existing basis, ensuring orthogonality is maintained.

Why is it important for P3 to be orthogonal to P0, P1, and P2?

Orthogonality ensures that the basis vectors are independent and that projections onto the subspace are simplified, which is essential for various applications like approximation and polynomial expansion.

Can P3 be explicitly expressed in terms of P0, P1, and P2?

Yes, P3 can often be expressed as a linear combination of higher-degree polynomials, orthogonalized against P0, P1, and P2, typically involving the Gram-Schmidt process.

What inner product is used to determine the orthogonality of the polynomials?

The inner product is usually defined as an integral over a specific interval with respect to a weight function, for example, ⟨f, g⟩ = ∫_a^b f(x)g(x)w(x) dx.

How does the choice of weight function affect the

orthogonality of the polynomials?

The weight function w(x) determines the inner product and thus influences which polynomials are orthogonal; different weight functions lead to different polynomial families like Legendre, Chebyshev, or Hermite polynomials.

What are some common polynomial families that arise from orthogonalization procedures?

Common families include Legendre, Chebyshev, Laguerre, and Hermite polynomials, each orthogonal under specific weight functions and intervals.

Why is establishing an orthogonal basis important in polynomial approximation?

An orthogonal basis simplifies calculations, minimizes approximation errors, and makes the process of projecting functions onto polynomial subspaces more efficient and stable.

Additional Resources

Find A Polynomial P_3 Such That $\{P_0, P_1, P_2, P_3\}$ Is An Orthogonal Basis For The Subspace

When working within the realm of polynomial spaces and orthogonality, a common and powerful task is to construct a sequence of polynomials that form an orthogonal basis for a given subspace. Specifically, finding a polynomial P_3 such that $\{P_0, P_1, P_2, P_3\}$ constitutes an orthogonal basis involves a systematic process rooted in the Gram-Schmidt orthogonalization procedure. This process ensures that each polynomial in the sequence is orthogonal to all preceding ones, which is essential for applications ranging from approximation theory to numerical analysis.

In this guide, we will explore step-by-step how to find such a polynomial P_3, assuming that P_0, P_1, and P_2 are known, and that they span a certain subspace of polynomials. We will also discuss the importance of orthogonal bases, how they are constructed, and the practical steps involved in deriving P_3.

Understanding the Context: Polynomial Spaces and Orthogonality

Polynomial Spaces and Subspaces

The space of all polynomials with real coefficients, typically denoted as \(\mathcal{P} \), is an infinite-dimensional vector space. When working within a specific subspace \(V \subset \mathcal{P} \), it is often useful to find a basis that simplifies computations—particularly an orthogonal basis.

Why Orthogonal Bases?

An orthogonal basis simplifies many operations:

- Projection: Computing the best approximation of a polynomial onto a subspace becomes straightforward.

- Integration and Inner Products: Calculations reduce to simple dot products.
- Numerical Stability: Orthogonal polynomials are numerically more stable for computations, especially in approximation problems like least squares.

Inner Product in Polynomial Spaces

The standard inner product for polynomials over an interval $[a, b]$ is often defined as:

$$
\langle f, g \rangle = \int_a^b f(x)g(x) w(x) \, dx
$$

where $w(x)$ is a weight function. For simplicity, many problems consider the weight function $w(x) = 1$, leading to the inner product:

$$
\langle f, g \rangle = \int_a^b f(x)g(x) \, dx
$$

Step-by-Step: Constructing P_3 Using Gram-Schmidt

Suppose we are given a set of polynomials P_0, P_1, P_2 that are orthogonal, or we are in the process of orthogonalizing them. Our goal is to find P_3 such that the set

$$
\{ P_0, P_1, P_2, P_3 \}
$$

forms an orthogonal basis for the subspace spanned by these four polynomials.

Step 1: Identify the Known Polynomials

Typically, these are:

- $P_0(x)$: The constant polynomial, often normalized to 1.
- $P_1(x)$: A linear polynomial, e.g., x.
- $P_2(x)$: A quadratic polynomial, e.g., x^2.

Suppose these are orthogonal (or orthogonalized) with respect to the chosen inner product.

Step 2: Define the Candidate Polynomial

Start with the polynomial $Q_3(x)$, which is the next polynomial in the sequence before orthogonalization. Usually, this is taken as:

$$
Q_3(x) = x^3
$$

or some other polynomial of degree 3, depending on the problem.

Step 3: Orthogonalize Q_3 Against P_0, P_1, P_2

Apply the Gram-Schmidt process:

$$
P_3(x) = Q_3(x) - \sum_{k=0}^{2} \frac{\langle Q_3, P_k \rangle}{\langle P_k, P_k \rangle} P_k(x)
$$

This ensures that P_3 is orthogonal to each of P_0, P_1, P_2.

Step 4: Calculate Inner Products

For each P_k, compute:

$$
\langle Q_3, P_k \rangle = \int_a^b Q_3(x) P_k(x) w(x) \, dx
$$
$$
\langle P_k, P_k \rangle = \int_a^b P_k(x)^2 w(x) \, dx
$$

which might involve standard integrals depending on the interval and weight function.

Step 5: Construct $P_3(x)$

Substitute the computed inner products into the orthogonalization formula:

$$
P_3(x) = Q_3(x) - \frac{\langle Q_3, P_0 \rangle}{\langle P_0, P_0 \rangle} P_0(x) - \frac{\langle Q_3, P_1 \rangle}{\langle P_1, P_1 \rangle} P_1(x) - \frac{\langle Q_3, P_2 \rangle}{\langle P_2, P_2 \rangle} P_2(x)
$$

This polynomial $P_3(x)$ will be orthogonal to all previous polynomials.

Practical Example: Constructing P_3 in Legendre Polynomial Space

To make the process concrete, consider the classical Legendre polynomials $\{ P_0, P_1, P_2, P_3 \}$ orthogonal over $[-1,1]$ with weight $w(x) = 1$.

Known Legendre Polynomials:

- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = \frac{1}{2}(3x^2 - 1)$
- $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

These are orthogonal by construction, but if starting from monomials, the Gram-Schmidt process recovers these polynomials.

Constructing P_3:

Suppose we start with $Q_3(x) = x^3$. To orthogonalize:

$$
P_3(x) = x^3 - \alpha P_0(x) - \beta P_1(x) - \gamma P_2(x)
$$

where

$$
\alpha = \frac{\langle x^3, P_0 \rangle}{\langle P_0, P_0 \rangle}
$$
$$
\beta = \frac{\langle x^3, P_1 \rangle}{\langle P_1, P_1 \rangle}
$$
$$
\gamma = \frac{\langle x^3, P_2 \rangle}{\langle P_2, P_2 \rangle}
$$

Calculating these inner products involves integrals over $[-1,1]$:

- $\langle x^3, 1 \rangle = \int_{-1}^1 x^3 dx = 0$
- $\langle x^3, x \rangle = \int_{-1}^1 x^4 dx = \frac{2}{5}$
- $\langle x^3, \frac{1}{2}(3x^2 - 1) \rangle$:

Compute numerator:

$$
\int_{-1}^1 x^3 \times \frac{1}{2}(3x^2 - 1) dx = \frac{1}{2} \int_{-1}^1 x^3 (3x^2 - 1) dx
$$

which simplifies and evaluates to zero, indicating orthogonality.

Finally, after computing the coefficients, the polynomial P_3 aligns with the known Legendre polynomial $P_3(x) = \frac{1}{2}(5x^3 - 3x)$.

Summary of the Process

To find a polynomial P_3 such that $\{ P_0, P_1, P_2, P_3 \}$ is an orthogonal basis:

1. Start with a candidate polynomial (e.g., x^3)
2. Orthogonalize against existing basis polynomials using the Gram-Schmidt process
3. Calculate inner products via integrals over the chosen interval
4. Subtract the projections to ensure orthogonality
5. Normalize if an orthonormal basis is desired

Additional Tips and Considerations

- Choosing the interval and weight function: The form of the orthogonal polynomials depends heavily on the interval $[a,b]$ and the weight $w(x)$. For classical polynomials, these are well-known.
- Normalization: For many applications, it's beneficial to normalize polynomials to have unit norm.
- Computational tools: Software like

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