Find The Value Of Y0 For Which The Solution Of The Initial Value Problem Y Y = 1 3 Sin T, Y(0) = Y0 Remains

Find The Value Of Y0 For Which The Solution Of The Initial Value Problem Y Y = 1 3 Sin T, Y(0) = Y0 Remains is a fundamental question in the study of differential equations, especially when analyzing the stability and long-term behavior of solutions to initial value problems (IVPs). Determining the specific initial condition Y0 that ensures the solution remains bounded, stable, or exhibits particular behavior is crucial in many applications across physics, engineering, and applied mathematics. In this article, we will explore the process of solving the initial value problem (IVP) involving a nonlinear differential equation, understand the implications of the initial condition, and develop a comprehensive approach to find the value of Y0 that guarantees the solution's desired properties.

Understanding the Differential Equation and Initial Conditions

The Differential Equation

The given differential equation appears to be:

$$Y \frac{dY}{dt} = \frac{1}{3} \sin t$$

This is a first-order nonlinear ordinary differential equation where Y is a function of t, and we are asked to find the initial value $Y_0 = Y(0)$ such that the solution remains stable or bounded.

The Initial Condition

The initial condition provided is:

$$ Y(0) = Y_0 $$

This condition specifies the starting point of the solution at $t=0$, which influences the entire trajectory of the solution $Y(t)$.

Transforming and Solving the Differential Equation

Step 1: Rewrite the Equation

The differential equation:

$$ Y \frac{dY}{dt} = \frac{1}{3} \sin t $$

can be rewritten as:

$$ Y dY = \frac{1}{3} \sin t \, dt $$

This separation of variables allows us to integrate both sides independently.

Step 2: Integrate Both Sides

Integrating the left side with respect to Y:

$$ \int Y \, dY = \frac{1}{2} Y^2 + C_1 $$

and the right side with respect to t:

$$ \int \frac{1}{3} \sin t \, dt = -\frac{1}{3} \cos t + C_2 $$

Combining the constants of integration $C = C_2 - C_1$, we get:

$$ \frac{1}{2} Y^2 = -\frac{1}{3} \cos t + C $$

or equivalently,

$$ Y^2 = -\frac{2}{3} \cos t + 2C $$

To determine C, apply the initial condition at $t=0$, where $Y(0) = Y_0$:

$$ Y_0^2 = -\frac{2}{3} \cos 0 + 2C $$

Since $\cos 0 = 1$, this simplifies to:

$$ Y_0^2 = -\frac{2}{3} \times 1 + 2C $$
$$ Y_0^2 = -\frac{2}{3} + 2C $$

Thus, the constant C is:

$$ C = \frac{1}{2} \left(Y_0^2 + \frac{2}{3} \right) $$

Expressing the General Solution

Putting everything together, the explicit solution for $Y(t)$ is:

$$ Y(t) = \pm \sqrt{ -\frac{2}{3} \cos t + 2C } $$

Substituting C:

$$ Y(t) = \pm \sqrt{ -\frac{2}{3} \cos t + Y_0^2 + \frac{2}{3} } $$

Simplify the expression inside the square root:

$$ Y(t) = \pm \sqrt{ Y_0^2 + \frac{2}{3} - \frac{2}{3} \cos t } $$

This is the general solution, contingent on the initial value Y_0.

Determining the Conditions for the Solution to Remain Bounded

Key Point: Boundedness of $Y(t)$

For the solution $Y(t)$ to remain real-valued and bounded over the interval $t \in \mathbb{R}$, the expression inside the square root must be non-negative:

$$ Y_0^2 + \frac{2}{3} - \frac{2}{3} \cos t \ge 0 \quad \forall t $$

Since $\cos t$ oscillates between -1 and 1, the minimal value of the expression occurs when $\cos t = 1$. Thus, the most restrictive condition is:

$$ Y_0^2 + \frac{2}{3} - \frac{2}{3} \times 1 \ge 0 $$
$$ Y_0^2 + \frac{2}{3} - \frac{2}{3} \ge 0 $$
$$ Y_0^2 \ge 0 $$

which is always true. The maximum value occurs when $\cos t = -1$:

$$ Y_0^2 + \frac{2}{3} - \frac{2}{3} \times (-1) = Y_0^2 + \frac{2}{3} + \frac{2}{3} = Y_0^2 + \frac{4}{3} $$

This is always positive for any real Y_0. Therefore, the expression inside the square root remains non-negative for all t, regardless of Y_0. However, to ensure the solution remains bounded and real-valued, the initial value Y_0 must be real.

Implication for Y_0

- Since the expression under the square root is always non-negative for any real Y_0, the solution remains real-valued for all t.
- To ensure the solution remains bounded (i.e., does not tend to infinity), Y_0 can be any real number.

Special Cases and Stability Analysis

Case 1: $Y_0 = 0$

If $Y_0 = 0$, the solution simplifies to:

$$ Y(t) = \pm \sqrt{ \frac{2}{3} - \frac{2}{3} \cos t } $$

Since $\cos t$ oscillates between -1 and 1, the solution oscillates between:

$$ Y(t) = 0 \quad \text{and} \quad \pm \sqrt{\frac{4}{3}} $$

which is bounded.

Case 2: Larger initial values Y_0

For larger initial values, the solution's amplitude increases, but it remains bounded as long as Y_0 is finite because the expression inside the square root remains bounded due to the oscillatory nature of $\cos t$.

Conclusion: Finding the Specific Y_O for Remained Solution

Based on the above analysis, the key takeaway is:

- The solution remains real and bounded for any real initial value Y_0.
- The initial value Y_0 directly influences the amplitude of the oscillating solution.
- To ensure the solution remains bounded, the initial value must be finite and real.

Therefore, the value of Y_0 that guarantees the solution remains bounded is any real number.

Summary of Key Points

- The differential equation $Y \frac{dY}{dt} = \frac{1}{3} \sin t$ can be integrated to find a general solution involving the initial condition Y_0.
- The solution:

$$ Y(t) = \pm \sqrt{ Y_0^2 + \frac{2}{3} - \frac{2}{3} \cos t } $$

- For the solution to remain real and bounded, the initial value Y_0 must be any real number.
- The behavior of the solution oscillates with amplitude depending on Y_0, but remains bounded for all real Y_0.

Final Remarks and Applications

Understanding the relationship between initial conditions and the behavior of solutions to nonlinear differential equations is critical in various scientific fields. Whether modeling physical systems, electrical circuits, or biological processes, the ability to determine initial values that lead to stable, bounded solutions informs both theoretical insights and practical applications.

In summary, for the initial value problem:

$$ Y \frac{dY}{dt} = \frac{1}{3} \sin t, \quad Y(0) = Y_0 $$

the solution remains bounded and real-valued for any real initial condition Y_0. The precise amplitude of oscillation depends on this initial value, but the overall boundedness is guaranteed across all real numbers.

Keywords for SEO Optimization:
- Find the value of Y_0

Frequently Asked Questions

What is the initial value problem given in the differential equation Y' = (1/3) sin t with Y(0) = Y0?

The initial value problem is $Y' = (1/3) \sin t$ with the initial condition $Y(0) = Y0$, where we seek the value of Y0 such that the solution remains finite or bounded.

How do we find the general solution for the differential equation Y' = (1/3) sin t?

Integrate both sides: $Y(t) = \int (1/3) \sin t \, dt + C = -(1/3) \cos t + C$, where C is the constant of integration.

What condition must be satisfied for the solution Y(t) to remain bounded as t approaches infinity?

Since the general solution is Y(t) = -(1/3) cos t + Y0, and cos t oscillates between -1 and 1, the solution remains bounded for all Y0; hence, Y0 does not affect boundedness.

Is the solution Y(t) = -(1/3) cos t + YO dependent on the initial value Y(O)?

Yes, the particular solution satisfying Y(0) = Y0 is Y(t) = -(1/3) cos t + Y0, where Y0 is determined by the initial condition.

For which value of YO does the solution remain bounded and well-defined?

Since the solution is always bounded regardless of Y0, any real value of Y0 makes the solution bounded; thus, the solution remains bounded for all Y0.

What is the significance of YO in the solution of the initial value problem?

Y0 represents the initial value of the solution at t = 0, and it shifts the entire solution vertically without affecting the boundedness or general form of the solution.

Additional Resources

Find The Value Of \(Y_0 \) For Which The Solution Of The Initial Value Problem \(\frac{dy}{dt} = \frac{1}{3} \sin t \), \(y(0) = y_0 \) Remains Bounded

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Introduction

In the study of differential equations, especially initial value problems (IVPs), a common question pertains to the behavior of solutions based on initial conditions. Specifically, understanding for which initial values y_0, the solution remains bounded over the entire domain (typically $t \in \mathbb{R}$) is crucial. This analysis reveals insights about the stability and long-term behavior of solutions.

The problem under consideration is:

$$
\frac{dy}{dt} = \frac{1}{3} \sin t, \quad y(0) = y_0
$$

Our goal is to find the particular value(s) of y_0 for which the solution $y(t)$ remains bounded as $t \to \infty$ (and $t \to -\infty$, if applicable).

Understanding the Differential Equation

The differential equation:

$$
\frac{dy}{dt} = \frac{1}{3} \sin t
$$

is a simple first-order linear ordinary differential equation with a sinusoidal forcing term. Its general solution can be obtained via direct integration, as it is already in a separable form.

Key Characteristics:

- The right-hand side $\frac{1}{3} \sin t$ is a bounded function since $|\sin t| \leq 1$.
- The solution will involve integrating a bounded function, leading to solutions that are potentially unbounded unless specific conditions are met.

Solving the Differential Equation

Step 1: Integrate both sides

Given:

$$
dy/dt = \frac{1}{3} \sin t
$$

integrate with respect to t:

$$
y(t) = \int \frac{1}{3} \sin t \, dt + C
$$

Step 2: Perform the integration

$$
\int \sin t \, dt = - \cos t + D
$$

Thus,

\[

$$
y(t) = - \frac{1}{3} \cos t + C
$$

where C is the constant of integration determined by initial conditions.

Step 3: Apply the initial condition

At $t = 0$:

$$
y(0) = y_0 = - \frac{1}{3} \cos 0 + C = - \frac{1}{3} (1) + C
$$

which simplifies to:

$$
C = y_0 + \frac{1}{3}
$$

Final solution:

$$
\boxed{
y(t) = - \frac{1}{3} \cos t + y_0 + \frac{1}{3}
}
$$

Analyzing the Solution's Behavior

The explicit solution:

$$
y(t) = y_0 + \frac{1}{3} - \frac{1}{3} \cos t
$$

offers a clear picture of the solution's behavior as t varies.

Key observations:

- The term $- \frac{1}{3} \cos t$ oscillates between $- \frac{1}{3}$ and $\frac{1}{3}$.
- The solution is bounded for all t, since $\cos t$ is bounded.

Explicit bounds:

$$
y(t) \in \left[y_0 + \frac{1}{3} - \frac{1}{3}, \quad y_0 + \frac{1}{3} + \frac{1}{3} \right] = \left[y_0, \quad y_0 + \frac{2}{3} \right]
$$

This indicates:

- The solution oscillates between y_0 and $y_0 + \frac{2}{3}$.
- The amplitude of oscillation is $\frac{1}{3}$.

When Does the Solution Remain Bounded?

Given the explicit form, the solution is inherently bounded for any initial value y_0 because the oscillatory part $- \frac{1}{3} \cos t$ is bounded.

Therefore:

$$
\boxed{
\text{For all } y_0 \in \mathbb{R}, \quad y(t) \text{ remains bounded.}
}
$$

This is a crucial conclusion: the solution is always bounded regardless of the initial condition.

Special Considerations and Interpretation

While the mathematical form indicates boundedness for all y_0, let's explore the implications:

- Uniqueness: The solution is unique for a given initial condition y_0.
- Long-term behavior: The solution's oscillation does not grow unboundedly; it stays within a fixed interval.
- Stability: The solution's boundedness is stable under the initial condition; changing y_0 shifts the solution vertically but does not affect the boundedness.

Contextualizing with Different Types of Differential Equations

The simplicity of the solution here stems from the nature of the differential equation:

- It is a linear, autonomous (though in this case, explicitly time-dependent) equation with a bounded forcing term.
- The solution involves an indefinite integral of a sinusoid, which is bounded and oscillatory.

Contrast with unbounded forcing or nonlinear equations:

- If the right-hand side were unbounded or grew without bound, solutions could potentially become unbounded.
- For nonlinear equations, the boundedness conditions are more intricate and depend on stability analysis.

In our case, the linearity and bounded forcing guarantee bounded solutions for all initial conditions.

Additional Insights

While the explicit solution indicates universal boundedness, some deeper points include:

- Periodic solutions: Since $y(t)$ involves $\cos t$, solutions are inherently periodic or oscillate periodically.
- Mean value of solutions: The average value over one period is $y_0 + \frac{1}{3}$, implying a shift depending on initial conditions.
- Implication for initial value y_0: Since the solution oscillates around $y_0 + \frac{1}{3}$, choosing y_0 affects the vertical shift but not the boundedness.

Summary of Findings

Aspect	Description
Differential Equation	$\frac{dy}{dt} = \frac{1}{3} \sin t$
General Solution	$y(t) = y_0 + \frac{1}{3} - \frac{1}{3} \cos t$
Boundedness	The solution remains bounded for all real y_0

Range of $y(t)$	Between y_0 and $y_0 + \frac{2}{3}$
Special Cases	No particular y_0 makes the solution unbounded

Final Conclusion

The key takeaway from this analysis is that:

> The solution $y(t)$ to the initial value problem $\frac{dy}{dt} = \frac{1}{3} \sin t$, $y(0) = y_0$ remains bounded for every real initial value y_0.

Therefore:

$$
\boxed{
\text{The value of } y_0 \text{ for which the solution remains bounded is any real number.}
}
$$

This conclusion highlights the stability and bounded oscillatory nature of solutions to linear differential equations with bounded sinusoidal forcing.

Additional Remarks and Broader Implications

- Physical Analogies: Many physical systems subjected to sinusoidal forcing, such as driven oscillators, exhibit bounded solutions unless nonlinearities or unbounded forcing terms induce resonance or unbounded growth.
- Mathematical Significance: The example underscores the importance of explicit solutions in

understanding qualitative behavior.

- Extension to Other Problems: For more complex or nonlinear equations, similar explicit solutions might not be available, necessitating qualitative analysis or stability theory.

References and Further Reading

- Differential Equations and Boundary Value Problems by C. Henry Edwards and David E. Penney
- Ordinary Differential Equations by E. L. Ince
- Stability Theory and its Applications in mathematical modeling
- Oscillatory Solutions and Boundedness in linear systems

Summary

In this in-depth exploration, we've demonstrated that the explicit solution to the initial value problem:

$$
\frac{dy}{dt} = \frac{1}{3} \sin t, \quad y(0) = y_0
$$

is:

$$
\boxed{
y(t) = y_0 + \frac{1}{3} - \frac{1}{3} \cos t
}
$$

which is inherently bounded for all real \(y_0 \) and all \(t \). Consequently, every initial condition \(y_0 \) leads to a bounded solution, and, therefore, there is no restriction on \(y_

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