

Give An Example Of A Linear Program For Which The Feasible Region Is Not Bounded, But The Optimal Objective

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Linear programming (LP) is a fundamental mathematical technique used in operations research, economics, engineering, and numerous other fields to optimize a linear objective function subject to a set of linear constraints. It allows decision-makers to find the best possible outcome—such as maximizing profit or minimizing cost—within a feasible region defined by these constraints.

A critical aspect of linear programming is understanding the nature of the feasible region—the set of all points that satisfy the constraints. Typically, the feasible region is a convex polyhedron, which can be bounded (forming a closed, finite region) or unbounded (extending infinitely in one or more directions). The boundedness of this region has significant implications for the existence and nature of optimal solutions.

In many cases, the feasible region is bounded, ensuring a finite optimal solution. However, intriguing scenarios exist where the feasible region is unbounded—meaning it extends infinitely in at least one direction—but the LP still has a finite, well-defined optimal value for the objective function. This phenomenon occurs because the objective function does not increase (or decrease) indefinitely along the unbounded directions, leading to a bounded optimal value despite an unbounded feasible region.

This article will explore this concept in detail, providing a concrete example of such a linear program, analyzing its feasible region, and explaining why the optimal solution remains finite despite unboundedness.

Understanding Bounded and Unbounded Feasible Regions in Linear Programming

What Is a Feasible Region?

The feasible region in a linear program is the set of all points that satisfy all the given constraints. These constraints are usually expressed as linear inequalities or equations, such as:

- $(a_1x_1 + a_2x_2 \leq b)$
- $(c_1x_1 + c_2x_2 = d)$

The intersection of these constraints forms a convex polyhedron in the decision variable space.

Bounded vs. Unbounded Regions

- Bounded Feasible Region: The region is closed and finite, meaning it contains all points within a finite volume. In this case, the optimal solution, if it exists, occurs at a vertex (corner point) of the region.
- Unbounded Feasible Region: The region extends infinitely in at least one direction. There are directions along which the feasible points can grow without limit.

Implications for Optimization

- For bounded regions, the LP has at least one optimal solution, which is finite and occurs at a vertex.
- For unbounded regions, the LP may have:
 - No finite optimal solution (if the objective improves infinitely along an unbounded direction).
 - A finite optimal solution (if the objective does not improve infinitely in any feasible direction).

This last point is crucial: an unbounded feasible region does not necessarily mean the LP's optimal value is infinite. The nature of the objective function and the constraints determines whether an optimal solution exists and whether it is bounded.

Scenario: Unbounded Feasible Region with Finite Optimal Objective

Let's consider the core question: Can a linear program have an unbounded feasible region but still possess a finite optimal value?

The answer is yes, and such cases are essential in understanding the subtleties of LP. This occurs when the objective function does not increase (or decrease) along the unbounded directions of the feasible region, effectively "capping" the optimal value.

Constructing an Example Linear Program

To illustrate this, we'll develop a specific LP example with the following characteristics:

- The feasible region is unbounded.
- The objective function has a finite maximum (or minimum).
- The potential unbounded directions do not improve the objective.

Example LP Formulation

Maximize:

$$[Z = 3x_1 + 2x_2]$$

Subject to constraints:

$$\begin{cases} x_1 - x_2 \leq 2 \\ x_1 + x_2 \geq 4 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases}$$

Analyzing the Constraints

Let's interpret the constraints:

1. $(x_1 - x_2 \leq 2)$: This restricts the difference between (x_1) and (x_2) .
2. $(x_1 + x_2 \geq 4)$: Ensures the sum of the variables is at least 4.
3. $(x_1 \geq 0)$ and $(x_2 \geq 0)$: Non-negativity constraints.

Graphical Representation and Feasible Region

Plotting these constraints in the (x_1, x_2) -plane:

- The line $(x_1 - x_2 = 2)$ divides the plane; feasible points satisfy $(x_1 - x_2 \leq 2)$, which is the region below or on this line.
- The line $(x_1 + x_2 = 4)$ divides the plane; feasible points satisfy $(x_1 + x_2 \geq 4)$, which is above or on this line.
- Non-negativity restricts us to the first quadrant.

The feasible region is the intersection of:

- The region below or on the line $(x_1 - x_2 = 2)$,
- The region above or on the line $(x_1 + x_2 = 4)$,
- The first quadrant $(x_1, x_2 \geq 0)$.

Resulting shape: An unbounded polygon extending infinitely in certain directions.

Identifying the Unbounded Directions

Let's analyze the boundaries:

- The intersection points of the lines:
- Solve for (x_1, x_2) :

$$\begin{cases} x_1 - x_2 = 2 \\ x_1 + x_2 = 4 \end{cases}$$

- Adding the equations:

$$2x_1 = 6 \implies x_1 = 3$$

\]

- Substituting back:

\[

$$3 + x_2 = 4 \implies x_2 = 1$$

\]

- So, the intersection point is $((3,1))$.

- The feasible region includes all points satisfying $(x_1 \geq 0, x_2 \geq 0)$, and lying between the lines $(x_1 - x_2 \leq 2)$ and $(x_1 + x_2 \geq 4)$.

- The corner point of the feasible region (the feasible vertex) is at $((3,1))$.

- The region extends infinitely in certain directions beyond the constraints, especially toward larger (x_1) and (x_2) .

Analyzing the Objective Function

The objective function is:

$$\[Z = 3x_1 + 2x_2 \]$$

- To maximize (Z) , consider the directions in which (Z) increases:

- Increasing (x_1) and (x_2) generally increases (Z) .

- But constraints limit how far we can go:

- The line $(x_1 - x_2 \leq 2)$ restricts the difference $(x_1 - x_2)$.

- The line $(x_1 + x_2 \geq 4)$ restricts the sum.

- Importantly, the feasible region is unbounded in the direction where both (x_1) and (x_2) grow large while remaining within the constraints.

Checking for Unboundedness in the Objective

- Does (Z) tend to infinity along some unbounded direction?
- To determine this, analyze the direction vectors along which the feasible region extends infinitely.
- For example, consider moving along the vector $(\mathbf{d}) = (d_1, d_2)$.
- Along $(\mathbf{d})$, the constraints become:

$$\begin{cases} (x_1 + t d_1) - (x_2 + t d_2) \leq 2 \\ (x_1 + t d_1) + (x_2 + t d_2) \geq 4 \\ x_1 + t d_1 \geq 0 \\ x_2 + t d_2 \geq 0 \end{cases}$$

- For unboundedness, we look for a direction $(\mathbf{d})$ such that:
- The constraints are satisfied as $(t \rightarrow \infty)$.
- The objective (Z) tends to infinity: $(Z(t) = 3(x_1 + t d_1) + 2(x_2 + t d_2) \rightarrow \infty)$.
- For the objective to tend to infinity, the components of

Frequently Asked Questions

What does it mean when a linear program has an unbounded feasible region but a bounded optimal objective value?

It means that while the set of feasible solutions extends infinitely in some direction, the objective function reaches its maximum (or minimum) at some boundary point, resulting in an optimal solution despite the unbounded feasible region.

Can you give an example of a linear program where the feasible region is unbounded but the objective function is bounded?

Yes. For instance, maximize $z = 2x + y$ subject to $x \geq 0$, $y \geq 0$, and $x - y \leq 1$. The feasible region is

unbounded in the positive x -direction, but the maximum z is achieved at $(1,0)$ with $z = 2(1) + 0 = 2$.

Why does a linear program with an unbounded feasible region sometimes still have a finite optimal value?

Because the objective function may attain its maximum or minimum at a specific boundary point within the feasible region, even if the feasible region extends infinitely in other directions.

How can we identify if a linear program's feasible region is unbounded but the optimal solution is finite?

By analyzing the constraints and the objective function's direction. If the feasible region extends infinitely but the objective function increases or decreases only up to a certain boundary, the optimal value is finite and occurs at that boundary.

What role do the constraints play in ensuring the optimal value remains bounded in an unbounded feasible region?

Constraints restrict the feasible region in certain directions, preventing the solution from extending infinitely in that direction for the objective function to grow unbounded, thus ensuring a finite optimal value.

Can the feasible region be unbounded in some variables but still have an optimal solution? How?

Yes. If the objective function is only affected by certain variables, and the feasible region is unbounded in variables not impacting the objective, the optimal solution can still be finite, occurring at a boundary where the objective is maximized.

Is it possible for a linear program to have multiple optimal solutions even if the feasible region is unbounded?

Yes. If the objective function is parallel to a boundary edge of the feasible region, there can be infinitely many optimal solutions along that edge, regardless of whether the feasible region is bounded or unbounded.

How does the concept of a 'bounded objective' differ from a 'bounded feasible region' in linear programming?

A 'bounded feasible region' means the entire set of feasible solutions is limited in extent, while a 'bounded objective' means the maximum or minimum value of the objective function is finite, which can occur even if the feasible region is unbounded.

What is a practical example where the feasible region is unbounded but the optimal solution is finite?

In diet problems where certain nutrients are limited, the feasible solutions can extend infinitely in some directions, but the cost or nutritional value (objective) reaches a minimum or maximum at a specific point, resulting in a finite optimal solution.

Additional Resources

Understanding Linear Programs with Unbounded Feasible Regions and Finite Optima

Linear programming (LP) is a powerful mathematical technique used for optimization, where the goal is to maximize or minimize a linear objective function subject to a set of linear constraints. While many LP problems have bounded feasible regions leading to finite optimal solutions, some interesting cases exist where the feasible region is unbounded, yet the optimal solution remains finite. This seemingly paradoxical situation warrants a detailed exploration, especially through concrete examples that illuminate the underlying principles.

Introduction to Feasible Regions and Optimality in Linear Programming

Before delving into specific examples, it's essential to clarify key concepts:

- Feasible Region: The set of all points in the decision variable space that satisfy all the problem's constraints.
- Bounded vs. Unbounded Region: A feasible region is bounded if it is contained within some finite space; otherwise, it is unbounded.
- Optimal Solution: The best (maximum or minimum) value of the objective function within the feasible region.

Typically, if the feasible region is bounded, the LP's optimal solution exists at a vertex (corner point). However, an unbounded feasible region might seem to suggest that the optimum could be infinite. But this is not always the case; certain problem structures can produce unbounded regions with finite optima.

Key Concepts: When Unbounded Feasible Regions Still Have Finite Optima

Understanding why an unbounded feasible region can have a finite optimal value involves these core ideas:

- Direction of Improvement: In LP, feasible directions are vectors along which the objective function improves without violating constraints.
- Recession Cone: The set of directions along which the feasible region extends infinitely. If the objective function improves indefinitely along this cone, the LP is unbounded.
- Bounded Objective Despite Unbounded Region: When the feasible region extends infinitely but the objective function's growth is limited in those directions, the optimal value remains finite.

In essence, the feasible region can be unbounded in certain directions, but if the objective's rate of increase in those directions is zero or negative, the LP will have a finite optimal solution.

Constructing an Example: The Linear Program

Let's craft a specific LP problem illustrating these concepts:

Objective:

Maximize $(z = 3x_1 + 2x_2)$

Subject to:

$$\begin{cases} x_1 - x_2 \leq 4 & \text{(Constraint 1)} \\ x_2 \geq 0 & \text{(Constraint 2)} \\ x_1 \text{ free} & \text{(Constraint 3)} \end{cases}$$

Interpreting the Constraints

- Constraint 1: Defines a boundary between the variables (x_1) and (x_2) . It limits the difference

between x_1 and x_2 , but doesn't impose an upper bound on x_1 directly.

- Constraint 2: Ensures x_2 is non-negative.

- Constraint 3: Indicates x_1 is unrestricted in sign (can be positive, negative, or zero).

Analyzing the Feasible Region

To understand the shape and properties of the feasible region, consider the constraints in detail:

- From Constraint 1, $x_1 \leq x_2 + 4$.

- From Constraint 2, $x_2 \geq 0$.

- Since x_1 is free, it can take on any real value, but it is limited above by $x_2 + 4$.

Now, examine the implications:

- For $x_2 = 0$, the maximum x_1 is 4.

- For $x_2 > 0$, $x_1 \leq x_2 + 4$. Since $x_2 \geq 0$, x_1 can be arbitrarily large in the negative direction (since it is free), but the upper bound depends on x_2 .

Key insight:

- Because x_1 is unrestricted in sign, the feasible region extends infinitely in the negative x_1 -direction.

- For a fixed $x_2 \geq 0$, x_1 can be any real number less than or equal to $x_2 + 4$. As x_2 increases, the upper bound on x_1 increases, but since x_1 is free downward, the feasible region is unbounded in the negative x_1 direction.

Graphically, the feasible region looks like an infinite strip extending infinitely to the left, bounded above by the line $x_1 = x_2 + 4$, with $x_2 \geq 0$.

Understanding the Objective Function's Behavior

Now, analyze how the objective function behaves within this feasible region:

$$z =$$

$$z = 3x_1 + 2x_2$$

\]

- To maximize (z) , we want (x_1) and (x_2) to be as large as possible.
- Since $(x_2 \geq 0)$, increasing (x_2) can improve the objective's value if (x_1) is also increased.
- But recall the constraint $(x_1 \leq x_2 + 4)$. To maximize (z) , choose (x_1) as large as possible within the feasible region, which is at the upper boundary $(x_1 = x_2 + 4)$.

Substitute into the objective:

\[

$$z = 3(x_2 + 4) + 2x_2 = 3x_2 + 12 + 2x_2 = 5x_2 + 12$$

\]

Since $(x_2 \geq 0)$, (z) increases as (x_2) increases. As $(x_2 \rightarrow \infty)$, $(z \rightarrow \infty)$.

This suggests the LP is unbounded — the objective can grow indefinitely by increasing (x_2) .

Refining the Example: Achieving a Finite Optimum Despite Unboundedness

The previous example indicates an unbounded LP with an infinite optimal value. To craft an example where the feasible region is unbounded but the optimal value is finite, modify the constraints as follows:

Modified LP:

Objective:

$$\text{Maximize } (z = 3x_1 + 2x_2)$$

Subject to:

\[

\begin{cases}

$$x_1 - x_2 \leq 4$$

$$x_2 \geq 0$$

$$x_1 \leq 10$$

\end{cases}

\]

Analysis of the Modified Constraints

- The third constraint, $x_1 \leq 10$, caps x_1 from above.
- The first constraint, $x_1 - x_2 \leq 4$, can be rearranged as $x_1 \leq x_2 + 4$.
- Combine these to identify the feasible region:

- Since $x_1 \leq 10$ and $x_1 \leq x_2 + 4$, the feasible x_1 satisfy:

$$x_1 \leq \min(10, x_2 + 4)$$

- With $x_2 \geq 0$, the maximum x_1 for a given x_2 is $x_1 = \min(10, x_2 + 4)$.
- As x_2 increases beyond 6, $x_2 + 4 \geq 10$, so the minimum becomes 10. Therefore:
- For $0 \leq x_2 \leq 6$, the upper bound is $x_2 + 4$, which is less than or equal to 10.
- For $x_2 \geq 6$, the upper bound is fixed at 10.

Feasible Region and Objective Analysis

- For $0 \leq x_2 \leq 6$: The maximum x_1 is $x_2 + 4$, so the feasible points are:

$$x_1 \leq x_2 + 4, \quad x_2 \geq 0, \quad x_1 \leq 10$$

But since $x_2 + 4 \leq 10$ for $x_2 \leq 6$, the maximum feasible x_1 is $x_2 + 4$.

- For $x_2 \geq 6$: The upper bound on x_1 is 10, which exceeds $x_2 + 4$. Since x_1

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