How That The Given Equation Is Not Exact But Becomes Exact When Multiplied By The Given Integrating Factor.

How That The Given Equation Is Not Exact But Becomes Exact When Multiplied By The Given Integrating Factor

In the study of differential equations, especially first-order equations, the concept of an exact equation plays a pivotal role in simplifying and solving problems efficiently. An exact differential equation is one that can be directly integrated by recognizing it as the total differential of some function. However, not all differential equations are exact in their initial form. Many such equations can be transformed into exact equations through the application of an integrating factor. This process involves multiplying the original differential equation by a carefully chosen function—called the integrating factor—that renders the equation exact. Understanding this transformation is fundamental to solving a broader class of differential equations systematically.

Understanding Exact Differential Equations

Definition of Exact Equations

An first-order differential equation can generally be written in the form:

```
M(x, y) dx + N(x, y) dy = 0
```

where M and N are functions of x and y. The equation is said to be exact if there exists a potential function *F(x, y)* such that:

```
dF = F_x dx + F_y dy = 0
```

which implies:

- **F_x = M(x, y)**
- **F_y = N(x, y)**

For the equation to be exact, the mixed partial derivatives must be equal, i.e.,

```
∂M/∂y = ∂N/∂x
```

Criteria for Exactness

Given M(x, y) and N(x, y), the equation is exact if:

1. The cross partial derivatives are equal:

$$\partial M/\partial y = \partial N/\partial x$$

If this condition holds, the equation can be integrated directly to find the potential function *F(x, y)*, leading to the solution.

Why Not All Equations Are Exact Initially?

Common Causes of Non-Exact Equations

Many differential equations do not satisfy the exactness condition inherently. Causes include:

- Coefficients M and N are not derivatives of a common potential function.
- The partial derivatives $\partial M/\partial y$ and $\partial N/\partial x$ are not equal.
- The equation involves functions that prevent straightforward integration.

Examples of Non-Exact Equations

Consider the equation:

$$(2xy + y^2)\,dx + (x^2 + 2xy)\,dy = 0$$

Calculating the partial derivatives:

- $\partial M/\partial y = 2x + 2y$
- $\partial N/\partial x = 2x + 2y$

Since these are equal, this particular equation is exact. But many similar equations do not satisfy this condition initially, requiring an integrating factor.

Introducing Integrating Factors

What Is an Integrating Factor?

An integrating factor is a function, typically denoted as $\mu(x, y)$, that when multiplied to the entire differential equation, makes it exact. It acts as a transformative tool to facilitate integration.

Purpose of the Integrating Factor

- Convert non-exact equations into exact ones.
- Allow for straightforward integration once the equation is exact.
- Expand the solvable set of differential equations.

Types of Integrating Factors

Depending on the nature of the original equation, integrating factors can be:

- Functions of x alone: $\mu(x)$
- Functions of y alone: $\mu(y)$
- Functions of both x and y: $\mu(x, y)$
  (less common and more complex)

Finding the Integrating Factor

Standard Methods for $\mu(x)$ or $\mu(y)$

Suppose the original equation is:

```
M(x, y) dx + N(x, y) dy = 0
```

and it is not exact. To find an integrating factor depending solely on x:

1. Compute:

$$(\partial N/\partial x - \partial M/\partial y) / M$$

If this ratio is a function of x alone, then:

$$\mu(x) = \exp\left(\int [(\partial N/\partial x - \partial M/\partial y) / M]\, dx\right)$$

Similarly, if the ratio is a function of y alone:

$$\mu(y) = \exp\left(\int [(\partial M/\partial y - \partial N/\partial x) / N]\, dy\right)$$

Example: Finding an Integrating Factor μ(x)

Given the equation:

$$(3x^2 y + y)\, dx + (x^3 + 2xy)\, dy = 0$$

Calculate the partial derivatives:

- $\partial M/\partial y = 3x^2 + 1$
- $\partial N/\partial x = 3x^2 + 2y$

Compute:

$$(\partial N/\partial x - \partial M/\partial y) / M = (3x^2 + 2y - 3x^2 - 1) / (3x^2 y + y) = (2y - 1) / y(3x^2 + 1)$$

If this ratio is a function of x alone, the method proceeds accordingly. Otherwise, consider other approaches or integrating factors depending on y or both variables.

Transforming the Equation into an Exact Form

Multiplying by the Integrating Factor

Once the appropriate integrating factor $\mu(x)$ or $\mu(y)$ is identified, multiply both sides of the differential equation:

$$\mu(x) M(x, y)\, dx + \mu(x) N(x, y)\, dy = 0$$

or

$$\mu(y) M(x, y)\, dx + \mu(y) N(x, y)\, dy = 0$$

This process ensures that the new coefficients satisfy the exactness condition:

$$\partial(\mu M)/\partial y = \partial(\mu N)/\partial x$$

Verifying Exactness After Multiplication

Check whether:

- $\partial(\mu M)/\partial y = \partial(\mu N)/\partial x$

holds true. If yes, the equation is now exact, and you can proceed to find the potential function *F(x, y)*.

Solving the Exact Equation

Integrating to Find the Potential Function

Once the equation is exact, follow these steps:

1. Integrate *μ M(x, y)* with respect to x:

$$F(x, y) = \int \mu M(x, y)\, dx + h(y)$$

where *h(y)* is an arbitrary function of y, determined by differentiating *F(x, y)* with respect to y and equating to *μ N(x, y)*.

2. Differentiate *F(x, y)* with respect to y:

$$\partial F/\partial y = \partial/\partial y \left[\int \mu M\, dx\right] + h'(y)$$

Set this equal to *μ N(x, y)* and solve for *h'(y)*. Integrate *h'(y)* to find *h(y)*.

3. Write the final solution as *F(x, y) = C*, where C is a constant.

Conclusion: The Power of Integrating Factors

The concept of an integrating factor is a cornerstone in solving first-order differential equations that are not initially exact. By multiplying the original

Frequently Asked Questions

What does it mean for a differential equation to be exact?

A differential equation is exact if it can be written in the form M(x, y) dx + N(x, y) dy = 0, where there exists a function F(x, y) such that dF = M dx + N dy. In other words, the equation represents the total differential of a potential function.

Why is an equation considered non-exact initially?

An equation is non-exact when the partial derivatives of M and N with respect to y and x, respectively, do not satisfy the condition $\partial M/\partial y = \partial N/\partial x$. This means it cannot be directly integrated to find a potential function.

How does an integrating factor convert a non-exact equation into an exact one?

An integrating factor is a function, often dependent on x or y, that when multiplied with the original differential equation, makes the modified equation exact by satisfying the condition for exactness.

What are common forms of integrating factors used to make an equation exact?

Common integrating factors include functions of x alone, y alone, or sometimes functions of both x and y. For example, $\mu(x)$, $\mu(y)$, or more complex functions depending on the form of the equation.

How do you determine the appropriate integrating factor for a given differential equation?

You analyze the given equation to see if it's close to exact and identify a function $\mu(x)$ or $\mu(y)$ that, when multiplied, satisfies the exactness condition. There are formulas and methods to test for these specific cases.

Can you give an example where a non-exact equation becomes exact after multiplying by an integrating factor?

Yes. For example, the equation (y - x) dx + (x + y) dy = 0 is not exact. Multiplying by $\mu(x) = 1/x$ makes it exact, allowing you to find a potential function and solve the equation.

Why is using an integrating factor a useful technique in solving differential equations?

Using an integrating factor simplifies solving non-exact differential equations by converting them into exact equations, which are easier to integrate and solve systematically.

Additional Resources

Understanding How a Non-Exact Differential Equation Becomes Exact with an Integrating Factor

When tackling differential equations, one encounters various forms and complexities. Among these, the concept of exact differential equations plays a crucial role in simplifying solutions. However, many differential equations initially appear non-exact, presenting a challenge for straightforward integration. The powerful tool to address this issue is the integrating factor, which transforms a non-exact equation into an exact one, thereby enabling direct integration and solution derivation. In this comprehensive review, we delve deeply into the mechanics of how a given differential equation, initially non-exact, can be rendered exact through multiplication by an appropriate integrating factor.

Understanding Exact Differential Equations

Before exploring how integrating factors modify a differential equation, it's essential to understand what defines an exact differential equation.

Definition of an Exact Differential Equation

A first-order differential equation of the form:

$$
M(x, y) \, dx + N(x, y) \, dy = 0
$$

is exact if there exists a potential function $\Psi(x, y)$ such that:

$$
\frac{\partial \Psi}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial \Psi}{\partial y} = N(x, y)
$$

In this case, the differential equation can be written as:

$$
d\Psi = 0
$$

which implies that $\Psi(x, y) = C$, a constant, defines the solution implicitly.

Criterion for Exactness

The key condition for exactness is the equality of mixed partial derivatives:

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If this condition holds, the differential equation is exact.

Why Do Some Equations Fail to Be Exact?

Not all differential equations satisfy the exactness condition. Many practical equations are non-exact, meaning:

$$
\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}
$$

This discrepancy indicates that there is no straightforward potential function Ψ that satisfies both partial derivatives simultaneously. Such equations pose a challenge because they cannot be directly integrated to find a potential function.

The Role of Integrating Factors

To overcome the obstacle posed by non-exactness, mathematicians introduce the concept of integrating factors.

What Is an Integrating Factor?

An integrating factor is a function $\mu(x, y)$ that, when multiplied with the original differential equation, converts it into an exact differential equation:

$$
\mu(x, y) \, M(x, y) \, dx + \mu(x, y) \, N(x, y) \, dy = 0
$$

The transformed equation now satisfies the exactness criterion, allowing for the straightforward integration to find the solution.

Purpose and Significance

- Converts non-exact equations into exact ones.

- Facilitates the use of potential functions to solve differential equations.
- Extends the class of solvable differential equations significantly.

How Does an Integrating Factor Make an Equation Exact?

The process by which an integrating factor renders an equation exact involves several steps and underlying principles.

Mathematical Explanation

Suppose the original differential equation:

$$
M(x, y) \, dx + N(x, y) \, dy = 0
$$

is not exact, i.e.,

$$
\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}
$$

Multiplying through by an integrating factor $\mu(x, y)$, we get:

$$
\mu M \, dx + \mu N \, dy = 0
$$

For this new equation to be exact, the following must hold:

$$
\frac{\partial (\mu M)}{\partial y} = \frac{\partial (\mu N)}{\partial x}
$$

Expanding these derivatives:

$$
\mu \frac{\partial M}{\partial y} + \frac{\partial \mu}{\partial y} M = \mu \frac{\partial N}{\partial x} + \frac{\partial \mu}{\partial x} N
$$

Rearranged as:

$$
\mu \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{\partial \mu}{\partial x} N - \frac{\partial \mu}{\partial y} M
$$

This differential relation guides the choice of $\mu(x, y)$. To find μ, one typically makes assumptions or looks for functions of x or y alone, leading to simplified conditions.

Key Idea

- The integrating factor μ is designed such that the modified functions μM and μN satisfy the exactness condition.
- The process involves solving auxiliary differential equations to identify μ.

Common Types of Integrating Factors

Depending on the form of M and N, integrating factors are often functions of:

- x alone: $\mu = \mu(x)$
- y alone: $\mu = \mu(y)$
- Both x and y: $\mu = \mu(x, y)$

The determination of which case applies depends on the structure of the original equation.

Integrating Factor as a Function of x Only

If the condition:

$$
\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0
$$

divided by N or M yields a function solely of x, then:

$$
\mu(x) = e^{\int P(x) \, dx}
$$

where $P(x)$ is derived from the structure of the differential equation.

Integrating Factor as a Function of y Only

Similarly, if:

$$
\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = Q(y)
$$

and the ratio of this difference to M or N results in a function solely of y, then:

$$
\mu(y) = e^{\int R(y) \, dy}
$$

Integrating Factor as a Function of x and y

In more complex cases, $\mu(x, y)$ may depend on both variables, requiring solving partial differential equations for μ.

Step-by-Step Process of Using an Integrating Factor

The typical approach involves:

1. Identify the form of the differential equation: Write it in the form $M(x, y) dx + N(x, y) dy = 0$.

2. Check for exactness: Verify whether $\frac{\partial M}{\partial y}$ equals $\frac{\partial N}{\partial x}$.

3. If not exact, determine an integrating factor:
- Check if $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{N}$ is a function of x only.
- Or verify if $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M}$ is a function of y only.
- If neither, attempt to find $\mu(x, y)$ via other methods.

4. Compute the integrating factor: Solve the relevant auxiliary differential equation to find μ.

5. Multiply the original equation by μ:
$$
\mu M \, dx + \mu N \, dy = 0
$$

6. Verify the new equation is exact: Check the derivatives.

7. Integrate to find the potential function $\Psi(x, y)$:
- Integrate μM with respect to x.
- Use the result to find Ψ, then differentiate to confirm consistency.

8. Write the implicit solution: $\Psi(x, y) = C$.

Practical Examples and Applications

Let's consider a practical illustration to solidify the understanding.

Example 1: A Non-Exact Equation and Its Integrating Factor

Suppose the differential equation:

$$
(2xy + y^2) dx + (x^2 + 2xy) dy = 0
$$

Check if it is exact:

$$
\frac{\partial M}{\partial y} = 2x + 2y
$$

$$
\frac{\partial N}{\partial x} = 2x + 2y
$$

Since both derivatives are equal, the equation is already exact. But to illustrate the role of an integrating factor, consider a modified version:

How That The Given Equation Is Not Exact But Becomes Exact When Multiplied By The Given Integrating Factor

How That The Given Equation Is Not Exact But Becomes Exact When Multiplied By The Given Integrating Factor

Related Articles

- QUESTION 25 You Are Testing The Null Hypothesis That There Is No Linear Relationship Between Two Variables,
- Keith Won A Cash Prize In An Art Competition. He Gave His Mother 2/5 Of The Money He

Won. Of The Remainder,

- Kyra Took Psychology In The Fall Semester And Is Now Taking Sociology. Several Of The Concepts Are Similar,