Let F(x,y) = X^2 - Xy + Y^2 -y. Find The Directions U And The Values Of D_u F(1,-1) For Which The Following

Let F(x,y) = X^2 - Xy + Y^2 - y. Find The Directions U And The Values Of D_u F(1,-1) For Which The Following

Understanding how functions behave with respect to their variables is fundamental in multivariable calculus. In particular, directional derivatives provide insights into how a function changes as we move in specific directions from a given point. This article explores the function $F(x, y) = x^2 - xy + y^2 - y$, focusing on determining the directions $\mathbf{u}$ along which the directional derivative at the point $(1, -1)$ attains specific values. We will systematically analyze the function, compute the gradient, and examine how to find directions and corresponding derivatives.

Understanding the Function $F(x, y) = x^2 - xy + y^2 - y$

Before diving into directional derivatives, it is essential to understand the behavior and properties of the function $F(x, y)$. This function combines quadratic and linear terms, making it a smooth, differentiable function suitable for gradient analysis.

Function Components and Behavior

The function consists of four terms:

- x^2: quadratic in x, always non-negative
- $-xy$: a bilinear term coupling x and y
- y^2: quadratic in y, always non-negative
- $- y$: linear in y

The combined effect of these terms creates a surface with potential maxima, minima, and saddle points, depending on the point of evaluation.

Significance of the Point $(1, -1)$

Evaluating the function and its derivatives at $(1, -1)$ offers insight into the local behavior of F near this point. Understanding the gradient at this point is crucial for analyzing directional derivatives.

Calculating the Gradient $\nabla F(x, y)$

The gradient vector points in the direction of the steepest ascent of the function. Its components are the partial derivatives with respect to x and y.

Partial Derivatives of F

Calculate the partial derivatives:

- $\frac{\partial F}{\partial x} = 2x - y$
- $\frac{\partial F}{\partial y} = -x + 2y - 1$

Gradient at $(1, -1)$

Plugging in $x = 1$, $y = -1$:

- $\frac{\partial F}{\partial x} (1, -1) = 2(1) - (-1) = 2 + 1 = 3$
- $\frac{\partial F}{\partial y} (1, -1) = -1 + 2(-1) - 1 = -1 - 2 - 1 = -4$

Therefore,

$$
\nabla F(1, -1) = \langle 3, -4 \rangle
$$

This vector indicates the direction of steepest ascent at $(1, -1)$.

Directional Derivatives and Their Computation

The directional derivative $D_{\mathbf{u}} F$ measures the rate of change of F in the direction of a unit vector $\mathbf{u}$.

Definition of the Directional Derivative

Given a unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$, the directional derivative of F at point (x, y) in the direction of $\mathbf{u}$ is:

$$
D_{\mathbf{u}} F(x, y) = \nabla F(x, y) \cdot \mathbf{u} = \frac{\partial F}{\partial x} u_1 + \frac{\partial F}{\partial y} u_2
$$

where $\mathbf{u}$ satisfies:

$$
u_1^2 + u_2^2 = 1
$$

Finding Directions $\mathbf{u}$ Corresponding to Specific $D_{\mathbf{u}} F(1, -1)$ Values

Suppose we want to find all directions $\mathbf{u}$ (unit vectors) such that the directional derivative at $(1, -1)$ equals a specific value α. The general form becomes:

$$
D_{\mathbf{u}} F(1, -1) = \nabla F(1, -1) \cdot \mathbf{u} = 3 u_1 - 4 u_2 = \alpha
$$

with the constraint:

$$
u_1^2 + u_2^2 = 1
$$

Method to Find $\mathbf{u}$ for a Given α

1. Express u_1 in terms of u_2:

\[

$$
u_1 = \frac{\alpha + 4 u_2}{3}
$$

2. Plug into the unit vector constraint:

$$
\left(\frac{\alpha + 4 u_2}{3} \right)^2 + u_2^2 = 1
$$

3. Simplify and solve for u_2:

$$
\frac{(\alpha + 4 u_2)^2}{9} + u_2^2 = 1
$$

$$
\frac{\alpha^2 + 8 \alpha u_2 + 16 u_2^2}{9} + u_2^2 = 1
$$

$$
\frac{\alpha^2 + 8 \alpha u_2 + 16 u_2^2}{9} + u_2^2 = 1
$$

Multiply through by 9:

$$
\alpha^2 + 8 \alpha u_2 + 16 u_2^2 + 9 u_2^2 = 9
$$

$$
\alpha^2 + 8 \alpha u_2 + 25 u_2^2 = 9
$$

This quadratic in u_2:

$$
25 u_2^2 + 8 \alpha u_2 + (\alpha^2 - 9) = 0
$$

4. Solve for u_2:

$$
u_2 = \frac{-8 \alpha \pm \sqrt{(8 \alpha)^2 - 4 \times 25 \times (\alpha^2 - 9)}}{2 \times 25}
$$

which simplifies to:

$$
u_2 = \frac{-8 \alpha \pm \sqrt{64 \alpha^2 - 100 (\alpha^2 - 9)}}{50}
$$

Further simplification:

$$
u_2 = \frac{-8 \alpha \pm \sqrt{64 \alpha^2 - 100 \alpha^2 + 900}}{50} = \frac{-8 \alpha \pm \sqrt{-36 \alpha^2 + 900}}{50}
$$

For real solutions, the discriminant must be non-negative:

$$
-36 \alpha^2 + 900 \geq 0 \Rightarrow 36 \alpha^2 \leq 900 \Rightarrow \alpha^2 \leq 25
$$

$$
\Rightarrow |\alpha| \leq 5
$$

Thus, for $|\alpha| \leq 5$, there exist directions $\mathbf{u}$ such that the directional derivative equals α.

Summary of Results and Interpretation

Based on the above calculations:

- The maximum directional derivative at $(1, -1)$ is in the direction of the gradient $\nabla F(1, -1) = \langle 3, -4 \rangle$, and its value is:

$$
\| \nabla F(1, -1) \| = \sqrt{3^2 + (-4)^2} = 5
$$

- The minimum directional derivative (steepest descent) is in the opposite direction:

$$
- \nabla F(1, -1) = \langle -3, 4 \rangle
$$

with value:

$$
-5
$$

- For any α between -5 and 5, solutions for $\mathbf{u}$ exist, and the corresponding directions can be explicitly computed as shown.

Practical Applications and Significance

Understanding directional derivatives and the directions in which the function increases or decreases most rapidly is crucial in various fields:

- Optimization: Finding maxima and minima involves analyzing the gradient and directional derivatives.
- Physics: Describing how physical quantities change in specific directions.
- Engineering: Sensitivity analysis in systems modeled by multivariable functions.
- Economics: Analyzing marginal effects in multiple variables.

Frequently Asked Questions

Given the function $F(x, y) = x^2 - xy + y^2 - y$, how do you find the gradient ∇F at the point $(1, -1)$?

The gradient ∇F is given by the partial derivatives: $F_x = 2x - y$, $F_y = -x + 2y - 1$. At $(1, -1)$, $F_x = 2(1) - (-1) = 2 + 1 = 3$, and $F_y = -1 + 2(-1) - 1 = -1 - 2 - 1 = -4$. So, $\nabla F(1, -1) = (3, -4)$.

How do you determine the direction vector U for which the directional derivative $D_u F(1, -1)$ is

maximized?

The directional derivative is maximized when the direction vector U is in the same direction as the gradient ∇F at the point. Therefore, U = (3, -4) normalized to a unit vector, which is (3/5, -4/5).

What is the maximum value of the directional derivative $D_u F(1, -1)$ at the point (1, -1)?

The maximum directional derivative equals the magnitude of the gradient at that point. The magnitude is $\sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$.

For which directions U does the directional derivative $D_u F(1, -1)$ equal zero?

The directional derivative is zero when U is orthogonal to the gradient ∇F. Since $\nabla F = (3, -4)$, the directions satisfy $3u_x - 4u_y = 0$, or $u_y = (3/4) u_x$. The unit vector in such directions can be taken as (4/5, 3/5) or any scalar multiple.

How can you compute the directional derivative D_u

F(1, -1) in an arbitrary direction U?

The directional derivative in the direction $U = (u_x, u_y)$ is given by $D_u F = \nabla F \cdot U = 3u_x - 4u_y$, where U is a unit vector. Ensure U is normalized before calculating to get the correct directional derivative.

Additional Resources

Gradient and Directional Derivatives: An In-Depth Exploration of $F(x, y) = x^2 - xy + y^2 - y$

Introduction

In the realm of multivariable calculus, understanding how a function behaves in different directions at a specific point is fundamental. Whether you're analyzing the steepness of a mountain or optimizing a complex system, the concepts of gradient and directional derivatives serve as powerful tools. Today, we will dissect the function:

\[

F(x, y) = x^2 - xy + y^2 - y
\]

focusing on how to determine the directions \(\mathbf{u} \) (unit vectors) and the corresponding rates of change \(D_{\mathbf{u}}F(1, -1) \) at the point \((1, -1) \).

Understanding the Function \(F(x, y) \)

Before diving into derivatives and directions, it’s essential to comprehend the structure of \(F \). The function combines quadratic and linear terms:

- \(x^2 \): a paraboloid opening upwards in the x-direction.
- \(-xy \): coupling the variables, introducing a saddle-like behavior.
- \(y^2 \): paraboloid component in y.
- \(- y \): a linear term shifting the function vertically.

This composition suggests that \(F \) exhibits a blend of convex and saddle features, making the analysis of its directional derivatives rich and nuanced.

The Gradient $\nabla F(x, y)$: The Key to Directional Derivatives

The gradient vector points in the direction of the steepest ascent of F at a point, with its magnitude corresponding to the rate of increase in that direction. It is calculated as:

$$
\nabla F(x, y) = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right)
$$

Calculating the partial derivatives:

- $\frac{\partial F}{\partial x} = 2x - y$
- $\frac{\partial F}{\partial y} = -x + 2y - 1$

Step 1: Find $\nabla F(1, -1)$

Plugging in the point $(x, y) = (1, -1)$:

\[

\begin{aligned}
\frac{\partial F}{\partial x}(1, -1) &= 2(1) - (-1) = 2 + 1 = 3 \\
\frac{\partial F}{\partial y}(1, -1) &= -1 + 2(-1) - 1 = -1 - 2 - 1 = -4
\end{aligned}
\]

Thus,

\[
\nabla F(1, -1) = (3, -4)
\]

This vector indicates that at \((1, -1) \), the steepest ascent points in the direction of \((3, -4) \).

Step 2: Normalizing the Gradient to Find \(\mathbf{u} \)

Since the direction \(\mathbf{u} \) must be a unit vector, we normalize the gradient:

\[
\mathbf{u} = \frac{\nabla F(1, -1)}{\|\nabla F(1, -1)\|} = \frac{(3, -4)}{\sqrt{3^2 + (-4)^2}} = \frac{(3, -4)}{\sqrt{9 + 16}} =

\frac{(3, -4)}{5}
\]

\[
\boxed{
\mathbf{u} = \left(\frac{3}{5}, -\frac{4}{5}
\right)
}
\]

This unit vector points in the direction of maximum increase of \(F \) at \((1, -1) \).

Step 3: Calculating the Directional Derivative \(D_{\mathbf{u}} F(1, -1) \)

The directional derivative of \(F \) at \((1, -1) \) in the direction \(\mathbf{u} \) is given by:

\[
D_{\mathbf{u}}F(1, -1) = \nabla F(1, -1) \cdot
\mathbf{u}
\]

Calculating:

\[

$$
D_{\mathbf{u}}F(1, -1) = (3, -4) \cdot \left(\frac{3}{5}, -\frac{4}{5} \right) = 3 \times \frac{3}{5} + (-4) \times \left(- \frac{4}{5}\right) = \frac{9}{5} + \frac{16}{5} = \frac{25}{5} = 5
$$

Interpretation: The directional derivative in the direction of the gradient vector (which points uphill) is 5, indicating a significant rate of increase.

Step 4: Finding All Directions $\mathbf{u}$ and Corresponding Derivatives

While the maximum rate of change aligns with the gradient, other directions are equally important in a comprehensive analysis.

General form of $D_{\mathbf{u}} F(1, -1)$:

$$
D_{\mathbf{u}} F(1, -1) = \nabla F(1, -1) \cdot \mathbf{u} = 3u_1 - 4u_2
$$

where $\mathbf{u} = (u_1, u_2)$, with the constraint:

$$
u_1^2 + u_2^2 = 1
$$

Step 5: Maximize and Minimize the Directional Derivative

Since $D_{\mathbf{u}}F(1, -1) = 3u_1 - 4u_2$, the maximum and minimum values occur when $\mathbf{u}$ is aligned or anti-aligned with $\nabla F(1, -1)$.

- Maximum: When $\mathbf{u}$ is in the same direction as ∇F:

$$
D_{\max} = \|\nabla F(1, -1)\| = 5
$$

- Minimum: When $\mathbf{u}$ is in the opposite direction:

$$
D_{\min} = -\|\nabla F(1, -1)\| = -5
$$

Explicitly, the directions are:

- Maximum rate of change: $\mathbf{u} = \left(\frac{3}{5}, -\frac{4}{5} \right)$
- Minimum rate of change: $\mathbf{u} = \left(-\frac{3}{5}, \frac{4}{5} \right)$

Summary of Key Findings

Quantity	Expression / Value	Interpretation
Gradient $\nabla F(1, -1)$	$(3, -4)$	Direction of steepest ascent
Unit vector $\mathbf{u}$ for maximum increase	$\left(\frac{3}{5}, -\frac{4}{5} \right)$	Direction of steepest ascent
Directional derivative at $(1, -1)$ in that direction	5	Rate of increase in that direction
Unit vector $\mathbf{u}$ for maximum decrease	$\left(-\frac{3}{5}, \frac{4}{5} \right)$	Direction of steepest descent
Directional derivative for that	-5	Rate of decrease

Implications and Applications

Understanding the gradient and directional derivatives of F at $(1, -1)$ offers deep insights into the function's local behavior:

- Optimization: To find maxima or minima, follow the gradient direction.
- Sensitivity analysis: The magnitude of the gradient indicates how sensitive F is to small changes around the point.
- Directional analysis: Knowing the directions of maximum and minimum change helps in navigation and decision-making in multivariable systems.

In practical scenarios—such as physics, engineering, or economics—these concepts facilitate precise control and prediction of system behavior.

Concluding Remarks

The analysis of $F(x, y) = x^2 - xy + y^2 - y$ at the point $(1, -1)$ exemplifies the power of calculus tools in dissecting complex functions. By calculating the gradient and normalizing it, we identified the exact directions for maximum and minimum rates of change, along with their values. This detailed exploration underscores the importance of understanding the geometry behind multivariable functions, transforming abstract mathematical concepts into actionable insights.

Whether you’re optimizing a design, modeling a phenomenon, or simply seeking a deeper grasp of multivariable calculus, mastering the interplay between gradients, directions, and derivatives is essential.

Let F X Y X 2 Xy Y 2 Y Find The Directions U And The Values Of D U F 1 1 For Which The Following

Let F X Y X 2 Xy Y 2 Y Find The Directions U And The Values Of D U F 1 1 For Which The Following

Related Articles

- The First Contact Dates Have Changed To Centre Align, By Defaultthey Will Aligna. Top Leftb. Bottom Rightc.
- Calculate The Equilibrium Constant For The Electrochemical Cell In Problem 10. Identify The Correct Answer.
- Find A Relationship Between X And Y Such That (x,y) Is Equidistant (the Same Distance) From The Two Points.