Solve The Given Differential Equation By Using An Appropriate Substitution. The DE Is A Bernoulli Equation.

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Differential equations are fundamental to understanding various phenomena in science, engineering, and mathematics. Among the different types of differential equations, Bernoulli equations hold a significant place due to their structured form, allowing for systematic approaches to find solutions. When faced with a Bernoulli differential equation, using the appropriate substitution can simplify the process, transforming a nonlinear equation into a linear one, making it easier to solve. This article provides a comprehensive guide on how to solve a Bernoulli differential equation through suitable substitution, elucidating the method step-by-step to enhance your understanding and problem-solving skills.

Understanding Bernoulli Differential Equations

Before diving into the solution method, it’s essential to understand what a Bernoulli differential equation is and how it differs from other types of differential equations.

Definition and Standard Form

A Bernoulli differential equation is a first-order nonlinear ordinary differential equation of the form:

$$\frac{dy}{dx} + P(x) y = Q(x) y^n$$

where:
- $P(x)$ and $Q(x)$ are continuous functions on a given interval.
- $n \neq 0, 1$ (the cases $n=0$ and $n=1$ reduce to linear differential equations).

This form combines a linear term $P(x) y$ and a nonlinear term $Q(x) y^n$, which makes it nonlinear for $n \neq 1$.

Significance of Bernoulli Equations

Bernoulli equations are important because:
- They appear in various real-world models, including growth and decay processes, thermodynamics, and population dynamics.
- They can be transformed into linear differential equations, enabling straightforward solution methods.
- Recognizing them allows you to apply a systematic substitution technique, simplifying the solution process.

Approach to Solving a Bernoulli Equation Using Substitution

The primary strategy for solving Bernoulli equations involves transforming the nonlinear equation into a linear form via a substitution. The general steps include:

Step 1: Write the Differential Equation in Standard Form

Ensure the equation is expressed as:

$$ \frac{dy}{dx} + P(x) y = Q(x) y^n $$

Identify the functions $P(x)$ and $Q(x)$, and the exponent n.

Step 2: Make the Appropriate Substitution

Define a new variable v to simplify the nonlinear term:

$$ v = y^{1 - n} $$

This substitution leverages the exponent n to convert the nonlinear differential equation into a linear one in terms of v.

Step 3: Compute $\frac{dv}{dx}$ in Terms of y and $\frac{dy}{dx}$

Differentiate $v = y^{1 - n}$ with respect to x:

$$ \frac{dv}{dx} = (1 - n) y^{-n} \frac{dy}{dx} $$

Rearranged to express $\frac{dy}{dx}$:

$$ \frac{dy}{dx} = \frac{1}{1 - n} y^{n} \frac{dv}{dx} $$

This step is crucial as it relates the derivatives of y and v.

Step 4: Substitute Back into the Original Equation

Replace y^{n} and $\frac{dy}{dx}$ in the original differential equation:

$$ \frac{1}{1 - n} y^{n} \frac{dv}{dx} + P(x) y = Q(x) y^{n} $$

Express y in terms of v:

$$ y = v^{\frac{1}{1 - n}} $$

But since $v = y^{1 - n}$, it can be manipulated accordingly.

Step 5: Simplify to Obtain a Linear Differential Equation in v

The resulting equation should be linear in v and $\frac{dv}{dx}$:

$$\frac{dv}{dx} + (1 - n) P(x) v = (1 - n) Q(x)$$

This linear form is much easier to solve using standard methods.

Step 6: Solve the Linear Differential Equation

Apply the integrating factor method to solve for v:

1. Compute the integrating factor:

$$\mu(x) = e^{\int (1 - n) P(x) dx}$$

2. Multiply the entire differential equation by $\mu(x)$:

$$\frac{d}{dx} [\mu(x) v] = (1 - n) Q(x) \mu(x)$$

3. Integrate both sides with respect to x:

$$\mu(x) v = \int (1 - n) Q(x) \mu(x) dx + C$$

where C is the constant of integration.

4. Solve for v:

$$v = \frac{1}{\mu(x)} \left(\int (1 - n) Q(x) \mu(x) dx + C \right)$$

Step 7: Back-Substitute to Find y

Recall the substitution $v = y^{1 - n}$, so:

$$y = v^{\frac{1}{1 - n}}$$

Plug in the expression for v to obtain the explicit solution for y.

Worked Example: Solving a Bernoulli Equation

Let's illustrate the method with an example:

Given Differential Equation:

$$ \frac{dy}{dx} + 2 y = 3 y^{2} $$

Step 1: Recognize the form:

$$ \frac{dy}{dx} + P(x) y = Q(x) y^{n} $$

where:
- $P(x) = 2$
- $Q(x) = 3$
- $n = 2$

Step 2: Make the substitution:

$$ v = y^{1 - n} = y^{1 - 2} = y^{-1} $$

which implies:

$$ y = v^{-1} $$

Step 3: Compute $\frac{dv}{dx}$:

$$ v = y^{-1} \Rightarrow \frac{dv}{dx} = - y^{-2} \frac{dy}{dx} $$

or:

$$ \frac{dy}{dx} = - y^{2} \frac{dv}{dx} $$

Step 4: Substitute into the original equation:

$$ - y^{2} \frac{dv}{dx} + 2 y = 3 y^{2} $$

Divide through by y^{2}:

$$ - \frac{dv}{dx} + 2 \frac{y}{y^{2}} = 3 $$

But $y / y^{2} = y^{-1} = v$, so:

$$ - \frac{dv}{dx} + 2 v = 3 $$

Rearranged as:

$$ \frac{dv}{dx} - 2 v = -3 $$

Step 5: Solve the linear differential equation:

$$ \frac{dv}{dx} - 2 v = -3 $$

Compute the integrating factor:

$$ \mu(x) = e^{\int -2 dx} = e^{-2x} $$

Multiply through:

$$ e^{-2x} \frac{dv}{dx} - 2 e^{-2x} v = -3 e^{-2x} $$

which simplifies to:

$$ \frac{d}{dx} [e^{-2x} v] = -3 e^{-2x} $$

Integrate both sides:

$$ e^{-2x} v = \int -3 e^{-2x} dx + C $$

Compute the integral:

$$ \int -3 e^{-2x} dx = -3 \times \left(- \frac{1}{2} e^{-2x} \right) + C' = \frac{3}{2} e^{-2x} + C' $$

Thus,

$$ e^{-2x} v = \frac{3}{2} e^{-2x} + C $$

Solve for v:

$$ v = \frac{3}{2} + C e^{2x} $$

Recall $y = v^{-1}$, so:

$$ y = \frac{1}{v} = \frac{1}{\frac{3}{2} + C e^{2x}} $$

This is the general solution to the original Bernoulli differential equation.

Additional Tips and Common Pitfalls

Successfully solving Bernoulli equations requires attention to detail. Here are some tips and common pitfalls:

Tips for Effective Solution

- Always verify the form of the differential equation to confirm it matches the Bernoulli type.
- Carefully perform the substitution $v = y^{1 - n}$ and differentiate correctly.
- When integrating, pay attention to the integral

Frequently Asked Questions

What is a Bernoulli differential equation?

A Bernoulli differential equation is a first-order differential equation of the form dy/dx + P(x)y = Q(x)y^n, where n ≠ 0 or 1. It is nonlinear but can be transformed into a linear differential equation using an appropriate substitution.

How do you identify a Bernoulli equation in a differential equation?

You identify a Bernoulli equation by checking if it can be written in the form dy/dx + P(x)y = Q(x)y^n, with P(x) and Q(x) as functions of x and n as a real number not equal to 0 or 1.

What substitution is used to solve a Bernoulli differential equation?

The standard substitution is v = y^{1 - n}. This transforms the nonlinear Bernoulli equation into a linear differential equation in terms of v, which can then be solved using integrating factors.

Can you provide an example of solving a Bernoulli equation using substitution?

Yes. For example, consider dy/dx + y = x y^2. Using v = y^{-1}, we rewrite the equation and solve the resulting linear differential equation for v, then back-substitute to find y.

What are the steps involved in solving a Bernoulli equation using substitution?

The steps are: (1) Write the equation in standard form, (2) Identify n and make the substitution v = y^{1 - n}, (3) Rewrite the differential equation in terms of v, (4) Solve the resulting linear differential equation for v, (5) Back-substitute to find y.

What are common pitfalls when solving Bernoulli equations?

Common pitfalls include incorrect substitution, forgetting to adjust the differential when substituting, and algebraic errors during simplification. It's important to carefully follow each step and verify the solution.

How do you verify the solution obtained from a Bernoulli differential equation?

You verify by substituting the solution y(x) back into the original differential equation and checking whether both sides are equal, ensuring the solution satisfies the initial or boundary conditions if provided.

Are Bernoulli equations always solvable using substitution?

Most Bernoulli equations are solvable through the substitution method. However, some may require additional techniques or be reducible to other types of differential equations, so always analyze the specific form carefully.

Additional Resources

Solving Differential Equations Using Substitution: A Deep Dive into Bernoulli's Equation

Differential equations are fundamental to understanding various phenomena across science, engineering, economics, and beyond. Among the myriad types of differential equations, Bernoulli's equation holds a special place due to its nonlinear nature and the elegance of its solution methodology. This article provides a comprehensive review of how to solve Bernoulli's differential equations through the strategic use of substitution techniques, offering detailed explanations, step-by-step procedures, and insightful analysis to deepen your understanding of this essential mathematical tool.

Understanding Differential Equations and Their Classification

Before delving into Bernoulli equations specifically, it is essential to establish a clear understanding of what differential equations are and how they are classified.

What Are Differential Equations?

A differential equation is an equation that involves an unknown function and its derivatives. These equations describe how a quantity changes concerning one or more variables. For example, the rate of population growth, the cooling of an object, or the motion of a particle can all be modeled using differential equations.

Mathematically, a differential equation can be expressed as:

$$
F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0
$$

where:

- x is the independent variable,
- y is the unknown function of x,
- $y', y'', \ldots, y^{(n)}$ are derivatives of y with respect to x.

Classification of Differential Equations

Differential equations are classified based on various criteria:

- Order: The highest derivative present. For example, if the highest derivative is y', it is a first-order differential equation.
- Linearity: Whether the equation is linear or nonlinear with respect to the unknown function and its derivatives.
- Type: Whether the equation is ordinary (ODE) or partial (PDE), depending on the number of independent variables.

In this context, we focus primarily on first-order equations, especially Bernoulli's equations, which are nonlinear but can be transformed into linear equations through substitution.

Introducing Bernoulli's Differential Equation

Definition and General Form

Bernoulli's differential equation is a specific type of first-order nonlinear ODE that takes the form:

$$
\frac{dy}{dx} + P(x) y = Q(x) y^{n}
$$

where:

- $P(x)$ and $Q(x)$ are continuous functions on an interval,
- n is a real number, not equal to 0 or 1 (since those cases are linear).

This equation was introduced by Jacob Bernoulli in the late 17th century and is notable because, despite its nonlinearity, it can be transformed into a linear differential equation with an appropriate substitution.

Special Cases of Bernoulli's Equation

- When $n = 0$, the equation reduces to a linear first-order ODE:

$$
\frac{dy}{dx} + P(x) y = Q(x)
$$

- When $n = 1$, the equation is also linear:

$$
\frac{dy}{dx} + P(x) y = Q(x) y
\Rightarrow
\frac{dy}{dx} + \left(P(x) - Q(x)\right) y = 0
$$

- For values of $n \neq 0, 1$, the equation remains nonlinear but can be transformed into a linear form via substitution.

Methodology for Solving Bernoulli’s Equation

The core technique for solving Bernoulli's equations involves using a substitution that linearizes the nonlinear term y^n. This transformation simplifies the problem to solving a linear first-order differential equation, for which well-established solution methods exist.

Step-by-Step Solution Process

1. Rewrite the Equation: Express the Bernoulli equation in standard form:

$$
\frac{dy}{dx} + P(x) y = Q(x) y^{n}
$$

2. Introduce the Substitution: To handle the nonlinear term y^{n}, set:

$$
z = y^{1 - n}
$$

This substitution is strategic because it simplifies the powers of y.

3. Differentiate z: Find $\frac{dz}{dx}$:

$$
z = y^{1 - n} \Rightarrow \frac{dz}{dx} = (1 - n) y^{-n} \frac{dy}{dx}
$$

4. Express $\frac{dy}{dx}$ in terms of z: Rearranged, this becomes:

$$
\frac{dy}{dx} = \frac{1}{1 - n} y^{n} \frac{dz}{dx}
$$

5. Substitute into the Original Equation: Replace y^{n} and $\frac{dy}{dx}$:

$$
\frac{1}{1 - n} y^{n} \frac{dz}{dx} + P(x) y = Q(x) y^{n}
$$

Since $y^{1 - n} = z$, then $y = z^{\frac{1}{1 - n}}$. Substituting, the equation simplifies to a linear differential equation in z:

$$
\frac{dz}{dx} + (1 - n) P(x) z = (1 - n) Q(x)
$$

6. Solve the Linear ODE in z: Use integrating factors or other standard techniques.

7. Back-Substitute to Find y: Once $z(x)$ is found, retrieve y via:

$$
y = z^{\frac{1}{1 - n}}
$$

Practical Example: Solving a Bernoulli Equation

To illustrate the procedure thoroughly, consider the differential equation:

$$
\frac{dy}{dx} + 2 y = 3 y^{2}
$$

This is a Bernoulli equation where:

- $P(x) = 2$,
- $Q(x) = 3$,
- $n = 2$.

Step 1: Rewrite in standard form (already in standard form).

Step 2: Substitution:

$$
z = y^{1 - 2} = y^{-1}
$$

Step 3: Compute $\frac{dz}{dx}$:

$$
\frac{dz}{dx} = - y^{-2} \frac{dy}{dx}
$$

Step 4: Express \(\frac{dy}{dx} \):

\[
\frac{dy}{dx} = - y^{2} \frac{dz}{dx}
\]

Step 5: Substitute into original:

\[
- y^{2} \frac{dz}{dx} + 2 y = 3 y^{2}
\]

Divide through by \(y^{2} \):

\[
- \frac{dz}{dx} + \frac{2 y}{y^{2}} = 3
\]

Note that \(y = z^{-1} \), so:

\[
- \frac{dz}{dx} + 2 z = 3
\]

Rearranged:

\[
\frac{dz}{dx} - 2 z = -3
\]

Step 6: Solve the linear ODE:

Using integrating factor:

\[
\mu(x) = e^{-2x}
\]

Multiply through:

\[
e^{-2x} \frac{dz}{dx} - 2 e^{-2x} z = -3 e^{-2x}
\]

Left side is derivative of \(z e^{-2x} \):

\[
\frac{d}{dx} \left(z e^{-2x} \right) = -3 e^{-2x}
\]

Integrate both sides:

$$
z e^{-2x} = \int -3 e^{-2x} dx + C
$$

Integral:

$$
\int e^{-2x} dx = - \frac{1}{2} e^{-2x}
$$

So,

$$
z e^{-2x} = -3 \left(- \frac{1}{2} e^{-2x} \right) + C = \frac{3}{2} e^{-2x} + C
$$

Solve for z:

$$
z = \frac{\frac{3}{2} e^{-2x} + C}{e^{-2x}} = \frac{3}{2} + C e^{2x}
$$

Recall $y = z^{-1}$:

$$
y = \frac{1}{z} = \frac{1}{\frac{3}{2} + C e^{2x}}
$$

This is the general solution to the original Bernoulli equation.

Applications and Significance of Bernoulli's Equation

Bernoulli's equations are not just theoretical constructs; they have practical applications across various fields:

- Fluid Dynamics: Bernoulli's principle explains the relationship between pressure, velocity, and height in flowing fluids.
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