3. Given Initial Value Problem Y" + 2y + 5y = 0 Y(0) =3 & (0) = 1 = (a) Solve The Initial Value Problem.

3. Given Initial Value Problem Y" + 2y + 5y = 0 Y(0) = 3 & Y'(0) = 1 = (a) Solve The Initial Value Problem.

Understanding how to solve initial value problems (IVPs) involving second-order linear differential equations is fundamental in applied mathematics, physics, and engineering. In this article, we will thoroughly analyze and solve the differential equation:

$$
Y'' + 2Y' + 5Y = 0
$$

with the initial conditions:

$$
Y(0) = 3, \quad Y'(0) = 1.
$$

This process involves several key steps: identifying the characteristic equation, solving for roots, constructing the general solution, and then applying initial conditions to find specific constants. Let's proceed systematically.

Understanding the Differential Equation

The given differential equation is:

$$
Y'' + 2Y' + 5Y = 0
$$

It is a homogeneous second-order linear differential equation with constant coefficients. These types of equations are common in modeling oscillatory systems, electrical circuits, mechanical vibrations, and more.

Step 1: Formulate the Characteristic Equation

To solve this differential equation, we assume solutions of the form:

$$
Y(t) = e^{rt}
$$

where r is a constant to be determined. Substituting into the differential equation:

$$
r^2 e^{rt} + 2 r e^{rt} + 5 e^{rt} = 0
$$

Divide through by e^{rt} (which is never zero):

$$
r^2 + 2r + 5 = 0
$$

This quadratic equation is called the characteristic equation.

Step 2: Solve the Characteristic Equation

The quadratic is:

$$
r^2 + 2r + 5 = 0
$$

Applying the quadratic formula:

$$
r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
$$

where $a = 1$, $b = 2$, $c = 5$.

Calculate discriminant:

$$
\Delta = b^2 - 4ac = 2^2 - 4 \times 1 \times 5 = 4 - 20 = -16
$$

Since $\Delta < 0$, roots are complex conjugates:

$$
r = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i
$$

Roots:

$$
r_1 = -1 + 2i, \quad r_2 = -1 - 2i
$$

Step 3: Write the General Solution

For complex roots $r = \alpha \pm \beta i$, the general solution is:

$$
Y(t) = e^{\alpha t} \left(C_1 \cos \beta t + C_2 \sin \beta t \right)
$$

Here, $\alpha = -1$, $\beta = 2$. So, the general solution becomes:

$$
\boxed{
Y(t) = e^{-t} \left(C_1 \cos 2t + C_2 \sin 2t \right)
}
$$

Step 4: Apply Initial Conditions to Find Constants

Given:

$$
Y(0) = 3, \quad Y'(0) = 1
$$

Evaluate $Y(0)$:

$$
Y(0) = e^{0} \left(C_1 \cos 0 + C_2 \sin 0 \right) = 1 \times (C_1 \times 1 + C_2 \times 0) = C_1
$$

So:

$$
C_1 = 3
$$

Find $Y'(t)$:

Differentiate $Y(t)$:

$$
Y(t) = e^{-t}(C_1 \cos 2t + C_2 \sin 2t)
$$

Apply the product rule:

$$
Y'(t) = \frac{d}{dt} \left[e^{-t} \right] (C_1 \cos 2t + C_2 \sin 2t) + e^{-t} \frac{d}{dt} (C_1 \cos 2t + C_2 \sin 2t)
$$

Compute derivatives:

$$
\frac{d}{dt} e^{-t} = -e^{-t}
$$

$$
\frac{d}{dt} (C_1 \cos 2t + C_2 \sin 2t) = -2 C_1 \sin 2t + 2 C_2 \cos 2t
$$

Putting it all together:

$$
Y'(t) = -e^{-t} (C_1 \cos 2t + C_2 \sin 2t) + e^{-t} (-2 C_1 \sin 2t + 2 C_2 \cos 2t)
$$

Factor out e^{-t}:

$$
Y'(t) = e^{-t} \left[- (C_1 \cos 2t + C_2 \sin 2t) + (-2 C_1 \sin 2t + 2 C_2 \cos 2t) \right]
$$

Simplify inside brackets:

$$
Y'(t) = e^{-t} \left[- C_1 \cos 2t - C_2 \sin 2t - 2 C_1 \sin 2t + 2 C_2 \cos 2t \right]
$$

Group like terms:

$$
Y'(t) = e^{-t} \left[(- C_1 \cos 2t + 2 C_2 \cos 2t) + (- C_2 \sin 2t - 2 C_1 \sin 2t) \right]
$$

$$
Y'(t) = e^{-t} \left[(2 C_2 - C_1) \cos 2t + (- C_2 - 2 C_1) \sin 2t \right]
$$

Evaluate at $t=0$:

$$
Y'(0) = e^{0} \left[(2 C_2 - C_1) \times 1 + (- C_2 - 2 C_1) \times 0 \right] = (2 C_2 - C_1)
$$

Given $Y'(0) = 1$ and $C_1 = 3$:

$$
1 = 2 C_2 - 3
$$

Solve for C_2:

$$
2 C_2 = 4 \implies C_2 = 2
$$

Final Solution

Substitute $C_1 = 3$ and $C_2 = 2$ into the general solution:

$$
\boxed{
Y(t) = e^{-t} \left(3 \cos 2t + 2 \sin 2t \right)
}
$$

This is the specific solution to the initial value problem.

Summary of the Solution Process

- Formulate the characteristic equation from the differential equation.
- Calculate the roots, which are complex conjugates, indicating oscillatory behavior with exponential decay.
- Write the general solution based on the roots.
- Apply initial conditions to solve for the constants C_1 and C_2.
- Present the final particular solution that satisfies the IVP.

Applications of Solving Such Differential Equations

Understanding how to solve equations like $Y'' + 2Y' + 5Y = 0$ with initial conditions is essential in various fields:

1. **Mechanical Vibrations:** Modeling damped harmonic oscillators, where the exponential decay and oscillatory nature mirror physical systems like suspension bridges or vehicle suspensions.
2. **Electrical Engineering:** Analyzing RLC circuits where voltage and current evolve over time with oscillatory damping.
3. **Control Systems:** Designing systems with desired transient responses by understanding natural frequencies and damping ratios.

Conclusion

In this article, we've demonstrated a comprehensive approach to solving the initial value problem for the second-order linear differential equation $Y'' + 2Y' + 5Y = 0$. By deriving the characteristic roots, constructing the general solution, and applying initial conditions, we obtained an explicit expression:

$$
Y(t) = e^{-t} \left(3 \cos 2t + 2 \sin 2t \right)
$$

This solution encapsulates the behavior of a damped oscillatory system, illustrating how initial conditions influence the specific

Frequently Asked Questions

What is the initial value problem given in the differential equation Y'' + 2Y' + 5Y = 0 with Y(0) = 3 and Y'(0) = 1?

The initial value problem is a second-order linear homogeneous differential equation with initial conditions $Y(0) = 3$ and $Y'(0) = 1$.

How do we solve the differential equation Y'' + 2Y' + 5Y = 0?

We solve it by finding the characteristic equation $r^2 + 2r + 5 = 0$, then determining its roots to write the general solution.

What are the roots of the characteristic equation r^2 + 2r + 5 = 0?

The roots are complex: $r = -1 \pm 2i$, found using the quadratic formula.

What is the general solution to the differential equation based on the roots?

The general solution is $Y(t) = e^{-t}(A \cos 2t + B \sin 2t)$.

How do we determine the constants A and B using initial conditions?

We substitute $t=0$ into the general solution and its derivative, then solve the resulting equations using $Y(0)=3$ and $Y'(0)=1$.

What is the specific solution to the initial value problem?

The specific solution is $Y(t) = e^{-t}(5 \cos 2t + 1/2 \sin 2t)$.

Why is it important to verify the solution after solving the IVP?

Verifying ensures that the solution satisfies both the differential equation and the initial conditions, confirming its correctness.

Are there any special techniques needed for solving this differential equation?

No special techniques are needed beyond solving the characteristic equation and applying initial conditions; it involves standard methods for linear homogeneous equations.

Additional Resources

Comprehensive Analysis and Solution to the Initial Value Problem \(y'' + 2y' + 5y = 0 \) with \(y(0) = 3 \) and \(y'(0) = 1 \)

Introduction

In the realm of differential equations, initial value problems (IVPs) are fundamental in modeling real-world phenomena where conditions are specified at a particular point, often time or space. The problem at hand involves a second-order linear homogeneous differential equation with constant coefficients:

\[
y'' + 2y' + 5y = 0,
\]

along with initial conditions:

\[
y(0) = 3, \quad y'(0) = 1.
\]

This particular problem exemplifies the classical approach to solving linear differential equations with constant coefficients, involving characteristic equations and initial value application to determine specific solutions.

Understanding the Differential Equation

Nature and Classification

The differential equation:

$$
y'' + 2y' + 5y = 0,
$$

is a second-order linear homogeneous differential equation with constant coefficients. Its general form is:

$$
a y'' + b y' + c y = 0,
$$

where:

- $a = 1$,
- $b = 2$,
- $c = 5$.

Such equations are fundamental in physics and engineering, modeling systems like damped oscillators, electrical circuits, and mechanical vibrations.

Significance of the Homogeneous Form

Being homogeneous implies that the solutions can be superimposed and scaled, leading to a general solution that encompasses all possible solutions to the differential equation. The initial conditions then specify which particular solution applies to the problem at hand.

Solution Strategy

The standard approach involves:

1. Formulating the characteristic equation associated with the differential equation.
2. Solving for roots of the characteristic equation.
3. Constructing the general solution based on root types.
4. Applying initial conditions to determine constants.

Let's explore each step in detail.

Step 1: Formulating the Characteristic Equation

For a second-order linear differential equation with constant coefficients:

$$
a y'' + b y' + c y = 0,
$$

the characteristic equation is:

$$
a r^2 + b r + c = 0.
$$

Substituting the given coefficients:

$$
r^2 + 2 r + 5 = 0.
$$

Step 2: Solving the Characteristic Equation

The quadratic formula:

$$
r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},
$$

applies here with $a=1, b=2, c=5$:

$$
r = \frac{-2 \pm \sqrt{(2)^2 - 4 \times 1 \times 5}}{2} = \frac{-2 \pm \sqrt{4 - 20}}{2} = \frac{-2 \pm \sqrt{-16}}{2}.
$$

Since the discriminant is negative, the roots are complex conjugates:

$$
r = \frac{-2 \pm 4i}{2} = -1 \pm 2i.
$$

Root Types:

- Complex conjugate roots \(r = \alpha \pm \beta i \),
- with \(\alpha = -1 \),
- \(\beta = 2 \).

Step 3: Constructing the General Solution

For roots \(r = \alpha \pm \beta i \), the general solution is:

\[
y(t) = e^{\alpha t} \left(C_1 \cos{\beta t} + C_2 \sin{\beta t} \right),
\]

where \(C_1 \) and \(C_2 \) are arbitrary constants.

Applying the values:

\[
\boxed{
\begin{aligned}
y(t) &= e^{-t} \left(C_1 \cos{2t} + C_2 \sin{2t} \right).
\end{aligned}
}
\]

Step 4: Deriving the First Derivative \(y'(t) \)

To implement initial conditions, we need \(y'(t) \):

\[
y(t) = e^{-t} \left(C_1 \cos{2t} + C_2 \sin{2t} \right).
\]

Differentiate using the product rule:

\[
y'(t) = \frac{d}{dt} \left[e^{-t} (C_1 \cos{2t} + C_2 \sin{2t}) \right].
\]

Applying the product rule:

$$
y'(t) = e^{-t} \frac{d}{dt} (C_1 \cos{2t} + C_2 \sin{2t}) - e^{-t} (C_1 \cos{2t} + C_2 \sin{2t}).
$$

Calculate the derivative inside:

$$
\frac{d}{dt} (C_1 \cos{2t} + C_2 \sin{2t}) = -2 C_1 \sin{2t} + 2 C_2 \cos{2t}.
$$

Thus,

$$
y'(t) = e^{-t} (-2 C_1 \sin{2t} + 2 C_2 \cos{2t}) - e^{-t} (C_1 \cos{2t} + C_2 \sin{2t}),
$$

which simplifies to:

$$
\boxed{
y'(t) = e^{-t} \left[-2 C_1 \sin{2t} + 2 C_2 \cos{2t} - C_1 \cos{2t} - C_2 \sin{2t} \right].
}
$$

Applying Initial Conditions

Given:

$$
y(0) = 3,
$$
$$
y'(0) = 1.
$$

Evaluate at $t=0$:

$$
y(0) = e^{0} (C_1 \cos 0 + C_2 \sin 0) = 1 \times (C_1 \times 1 + C_2 \times 0) = C_1.
$$

Hence,

$$
C_1 = 3.
$$

Next, evaluate $y'(0)$:

$$
y'(0) = e^{0} \left[-2 C_1 \sin 0 + 2 C_2 \cos 0 - C_1 \cos 0 - C_2 \sin 0 \right].
$$

Simplify:

$$
y'(0) = 1 \times (0 + 2 C_2 \times 1 - 3 \times 1 - 0) = 2 C_2 - 3.
$$

Set equal to 1:

$$
2 C_2 - 3 = 1 \Rightarrow 2 C_2 = 4 \Rightarrow C_2 = 2.
$$

Final Solution

Substituting $C_1 = 3$ and $C_2 = 2$ into the general solution:

$$
\boxed{
y(t) = e^{-t} \left(3 \cos 2t + 2 \sin 2t \right).
}
$$

This function satisfies both the differential equation and the initial conditions.

Discussion and Interpretation

Behavior of the Solution

- The exponential decay term e^{-t} indicates the solution diminishes

over time, characteristic of a damped oscillator.
- The oscillatory component $\cos 2t$ and $\sin 2t$ with frequency 2 reflect harmonic motion.
- The initial amplitude is set by the constants C_1 and C_2, while the decay rate is governed by the exponential factor.

Physical Implications

In physical systems:

- The damping coefficient $\alpha = -1$ signifies a damping force proportional to velocity, reducing motion over time.
- The frequency $\beta = 2$ indicates how rapidly the system oscillates before damping dominates.
- The initial conditions specify the starting position and velocity, anchoring the solution to a real-world scenario.

Mathematical Significance

- The solution exemplifies the method of characteristic equations for constant-coefficient linear differential equations.
- The use of complex roots simplifies the process, especially when roots are non-real.
- The approach illustrates the importance of initial conditions in transforming the general solution into a particular one.

Extensions and Additional Insights

Alternative Methods

While the characteristic equation method is most straightforward here, other techniques include:

- Laplace Transforms: Useful for more complex initial conditions or nonhomogeneous equations.
- Series Solutions: Applicable if roots are repeated or complex roots are problematic.
- Numerical Methods:

3 Given Initial Value Problem Y 2y 5y 0 Y 0 3 Amp 0 1 A Solve The Initial Value Problem

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