For A Normal Population With Known Variance, S 2 , What Is The Confidence Level For The Confidence Interval.

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Understanding the confidence level associated with a confidence interval is fundamental in statistics, especially when estimating population parameters such as the mean. When dealing with a normal population where the variance (or its estimate) is known, the process of constructing a confidence interval becomes more straightforward. This article aims to clarify the concept of confidence levels in this context, exploring the underlying theory, calculations, and practical implications.

Introduction to Confidence Intervals in Normal Populations

Confidence intervals are statistical tools used to estimate an unknown population parameter, such as the mean, with a certain degree of confidence. They provide a range of plausible values based on sample data, emphasizing the uncertainty inherent in sampling.

In the case of a normal population with known variance, the construction of confidence intervals simplifies because the sampling distribution of the sample mean is well-understood and follows a normal distribution. This allows for precise calculation of the interval and its associated confidence level.

Key Concepts:

- Population Mean (μ): The average value of a variable in the population.
- Sample Mean ($\bar{x}$): The average value in a sample.
- Known Variance (σ^2): The population variance is known and fixed.
- Standard Deviation (σ): The square root of the variance.
- Confidence Level (1 - α): The probability that the true population parameter lies within the confidence interval.

Theoretical Foundation of Confidence Levels with Known Variance

When the population variance is known, and the population is normally distributed, the sampling distribution of the sample mean $\bar{x}$ is also normal, with mean μ and variance σ^2/n, where n is the sample size.

Constructing the Confidence Interval:

The general formula for the confidence interval for the population mean when variance is known is:

$$CI = \bar{x} \pm Z_{\alpha/2} (\sigma / \sqrt{n})$$

Where:

- $\bar{x}$: Sample mean
- σ: Known population standard deviation
- n: Sample size
- $Z_{\alpha/2}$: Critical value from the standard normal distribution corresponding to the desired confidence level

Understanding the Critical Value ($Z_{\alpha/2}$):

- The critical value $Z_{\alpha/2}$ is determined by the desired confidence level.
- For example, for a 95% confidence level, $\alpha = 0.05$, and $Z_{\alpha/2} \approx 1.96$.
- The value $Z_{\alpha/2}$ cuts off the upper $\alpha/2$ tail in the standard normal distribution, ensuring the specified confidence level.

Confidence Level and Its Connection to the Critical Value:

- The confidence level (e.g., 90%, 95%, 99%) indicates the proportion of such intervals, constructed from repeated samples, that would contain the true population mean.
- It directly relates to the critical value $Z_{\alpha/2}$, which determines the width of the interval.

Determining the Confidence Level for a Known Variance

Given a sample, the main question is: "What is the confidence level associated with a specific confidence interval?" When the variance is known, and the interval is constructed, the confidence level is connected to the probabilities under the standard normal distribution.

Step-by-Step Process:

1. Calculate the Sample Mean ($\bar{x}$): Obtain this from your data.
2. Identify the Known Variance or Standard Deviation (σ): Usually provided.
3. Determine the Sample Size (n): From your data collection.
4. Choose or Observe the Interval: The interval bounds are typically given or calculated.
5. Find the Critical Value ($Z\alpha/2$):: Based on the interval bounds and the normal distribution.

Calculating the Confidence Level:

- The confidence level is the probability that the standard normal variable Z falls within the critical bounds:

$$P(-Z\alpha/2 \leq Z \leq Z\alpha/2) = 1 - \alpha$$

- Expressed in terms of the sample data:

$$P(\bar{x} - Z\alpha/2 (\sigma/\sqrt{n}) \leq \mu \leq \bar{x} + Z\alpha/2 (\sigma/\sqrt{n})) = 1 - \alpha$$

- Rearranged, the confidence level is:

$$\text{Confidence Level} = P(|Z| \leq Z\alpha/2)$$

Practical Example: Calculating the Confidence Level

Suppose you have a sample with:

- Sample size $n = 50$
- Sample mean $\bar{x} = 100$
- Known population standard deviation $\sigma = 15$
- Confidence interval: (95, 105)

Step 1: Calculate the margin of error:

$$ME = (105 - 100) = 5$$

Step 2: Find the critical value $Z\alpha/2$:

$$Z\alpha/2 = ME / (\sigma / \sqrt{n}) = 5 / (15 / \sqrt{50}) \approx 5 / (15 / 7.07) \approx 5 / 2.12 \approx 2.36$$

Step 3: Determine the confidence level:

- The probability that Z falls between -2.36 and 2.36 is:

$$P(-2.36 \leq Z \leq 2.36) \approx 2 \Phi(2.36) - 1$$

Using standard normal distribution tables or calculator:

- $\Phi(2.36) \approx 0.9909$

Therefore,

- Confidence level ≈ 2 0.9909 - 1 = 0.9818 or 98.18%

Conclusion:

- The constructed interval corresponds approximately to a 98.2% confidence level.

This example illustrates how the confidence level relates to the critical value derived from the interval bounds.

Relationship Between Confidence Level, Significance Level, and Critical Values

Understanding the interplay between these concepts is crucial for accurate interpretation and construction of confidence intervals.

Definitions:

- Confidence Level ($1 - \alpha$): The proportion of intervals that will contain the true parameter over many repetitions.
- Significance Level (α): The probability of the interval not containing the true parameter.
- Critical Value ($Z_{\alpha/2}$): The standard normal value that corresponds to the tail probability $\alpha/2$.

Key Relationships:

- Increasing the confidence level (e.g., from 95% to 99%) increases the critical value $Z_{\alpha/2}$, resulting in a wider interval.
- Conversely, decreasing the confidence level narrows the interval but reduces the probability that it contains the true mean.

Summary Table:

Confidence Level	α (Significance Level)	Zα/2	Approximate Critical Values

90%	0.10	1.645	±1.645
95%	0.05	1.960	±1.960
99%	0.01	2.576	±2.576

Implications in Practice

In practical data analysis, knowing the confidence level associated with a confidence interval is essential for making informed decisions.

Important Points:

- When the population variance is known, the confidence level is directly determined by the critical value $Z\alpha/2$.
- The researcher can select a desired confidence level beforehand, then compute the interval accordingly.
- Alternatively, given an interval, one can determine the confidence level, as shown in the examples above.

Applications:

- Quality control in manufacturing
- Estimation in scientific experiments
- Financial risk assessment
- Policy making based on survey data

Limitations and Considerations

While the approach for known variance is straightforward, there are limitations and considerations to keep in mind:

- Assumption of Normality: The population must be normally distributed, especially for small sample sizes.
- Known Variance: In practice, the population variance is rarely known; typically, it is estimated from the sample, leading to the t-distribution.
- Sample Size: Larger samples tend to produce more reliable confidence intervals.
- Interval Interpretation: The confidence level does not imply that the specific interval calculated has a certain probability of containing the true mean; rather, it reflects the long-run frequency of such intervals.

Conclusion

Determining the confidence level for a confidence interval in a normal population

Frequently Asked Questions

What is the confidence level associated with a confidence interval when the population variance is known?

The confidence level indicates the proportion of such intervals that would contain the true population mean if the process were repeated multiple times. When the population variance is known, the confidence level is typically expressed as a percentage, such as 95%, representing the probability that the interval captures the true mean.

How is the confidence level determined in a normal population with known variance?

The confidence level is determined by the chosen significance level (alpha) and the corresponding critical value from the standard normal distribution (Z-distribution). For example, a 95% confidence level corresponds to alpha = 0.05 and uses critical values of ±1.96.

Does knowing the population variance impact the confidence level of the interval?

Knowing the population variance allows for the use of the Z-distribution to construct the confidence interval, which directly influences the calculation of the interval and the associated confidence level. It ensures more precise intervals compared to when variance is unknown.

What is the typical confidence level used in practice for normal populations with known variance?

The most commonly used confidence level in practice is 95%, but levels such as 90%, 99%, and others are also used depending on the desired precision and context.

How does the confidence level relate to the Z-score in the context of known variance?

The confidence level determines the Z-score (critical value) used in the calculation of the confidence interval. For example, a 95% confidence level corresponds to a Z-score of approximately 1.96, which defines the interval bounds.

Can the confidence level be adjusted in a normal population with known variance?

Yes, the confidence level can be adjusted by selecting a different significance level (alpha). Increasing the confidence level (e.g., from 95% to 99%) results in a wider interval, while decreasing it narrows the interval.

What role does the sample size play in determining the confidence level when variance is known?

While the confidence level is primarily set by the chosen alpha and Z-score, larger sample sizes lead to narrower confidence intervals for the same confidence level, increasing the precision of the estimate without changing the confidence level itself.

Is the confidence level fixed once the population variance is known?

No, the confidence level is not fixed by knowing the population variance; it is selected by the researcher based on how confident they want to be that the interval contains the true mean. The known variance helps in accurately calculating the interval for the chosen confidence level.

How does the known variance simplify the calculation of the confidence interval and its confidence level?

Knowing the population variance allows the use of the standard normal distribution (Z-distribution) for constructing the confidence interval, simplifying calculations and making the confidence level directly related to the Z-score without needing to estimate variance from the sample.

Additional Resources

Confidence Level for the Confidence Interval When Variance Is Known in a Normal Population

Understanding the confidence level associated with a confidence interval is fundamental in statistical inference, especially when working with normally distributed populations where the population variance is known. This comprehensive review aims to elucidate the concept, derivation, and implications of the confidence level in such contexts, providing clarity for statisticians, data analysts, and students alike.

Introduction to Confidence Intervals in Normal Populations

Confidence intervals (CIs) serve as a range of plausible values for an unknown population parameter, such as the mean. They are constructed based on sample data and provide an estimate of the precision of the sample statistic.

- Purpose of Confidence Intervals: To quantify uncertainty about the true population parameter.
- Key Components:
- Sample statistic (e.g., sample mean)
- Margin of error
- Confidence level (e.g., 95%, 99%)

In the context of a normal population with a known variance, the process simplifies due to the properties of the normal distribution, facilitating precise calculation of the confidence interval and its associated confidence level.

Preliminaries and Assumptions

Before delving into the derivation and properties, it’s important to understand the foundational assumptions:

- The population is normally distributed: $X \sim N(\mu, \sigma^2)$.
- The population variance σ^2 (and therefore the standard deviation σ) is known.
- A random sample $\{X_1, X_2, ..., X_n\}$ is obtained from the population.

These assumptions allow the use of the standard normal distribution to construct the confidence interval for the mean μ.

Constructing the Confidence Interval with Known Variance

Given the sample data:

- Sample mean: $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$
- Known population standard deviation: σ

The confidence interval for the population mean μ is:

$$
\boxed{
\bar{X} \pm Z_{\alpha/2} \times \frac{\sigma}{\sqrt{n}}
}
$$

where:

- $Z_{\alpha/2}$ is the critical value from the standard normal distribution corresponding to the desired confidence level.
- n is the sample size.

This interval is derived from the fact that:

$$
Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim N(0,1)
$$

Understanding the Confidence Level

Definition: The confidence level (denoted as $1 - \alpha$) is the proportion of such confidence intervals, constructed using the same sampling method, that would contain the true population mean μ if the process were repeated infinitely many times.

- For example, a 95% confidence level indicates that in the long run, 95% of the constructed confidence intervals will contain the true mean.

Key Point: The confidence level is a property of the interval construction method, not of any single interval. It reflects the procedure's reliability over repeated sampling.

Calculating the Critical Value $Z_{\alpha/2}$

The value $Z_{\alpha/2}$ is vital because it determines the width of the confidence interval for a given confidence level.

- For a chosen confidence level $1 - \alpha$:
- $Z_{\alpha/2}$ is the value such that:

$$
P\left(-Z_{\alpha/2} \leq Z \leq Z_{\alpha/2}\right) = 1 - \alpha
$$

- Common values:
- 90% confidence level: $Z_{0.05} \approx 1.645$
- 95% confidence level: $Z_{0.025} \approx 1.96$
- 99% confidence level: $Z_{0.005} \approx 2.576$

These values are obtained from the standard normal distribution table.

Relationship Between Confidence Level and Significance Level α

- The confidence level $1 - \alpha$ is directly related to the significance level α:

$$
\text{Confidence Level} = 1 - \alpha
$$

- The critical value $Z_{\alpha/2}$ corresponds to the upper $\alpha/2$ tail probability of the standard normal distribution.

Interpretation:
If you set a confidence level of 95%, then:

$$
\alpha = 1 - 0.95 = 0.05
$$
$$
\Rightarrow \alpha/2 = 0.025
$$
$$
\Rightarrow Z_{0.025} \approx 1.96
$$

This means that the probability that the standardized sample mean falls outside the interval $\bar{X} \pm Z_{0.025} \times \frac{\sigma}{\sqrt{n}}$ is 5%.

Implication of Known Variance for Confidence Level Determination

When the variance σ^2 is known:

- The construction of the confidence interval is straightforward.
- The critical value $Z_{\alpha/2}$ is directly used to determine the confidence level.
- The confidence level is explicitly controlled by selecting $Z_{\alpha/2}$, which corresponds to the desired long-run coverage probability.

This contrasts with situations where the variance is unknown, requiring the use of the Student’s t-distribution, which introduces additional considerations.

Practical Steps to Determine the Confidence Level

1. Identify the desired confidence level: e.g., 95%, 99%, etc.
2. Find the corresponding $Z_{\alpha/2}$: from standard normal tables or statistical software.
3. Calculate the confidence interval:

$$
\left[\bar{X} - Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}, \quad \bar{X} + Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right]
$$

4. Interpret the confidence level: as the proportion of such intervals that will contain the true mean if the process is repeated many times.

Understanding the Confidence Level in Context

- Long-Run Frequency: The confidence level is a statement about the procedure’s performance over many repetitions, not about any single calculated interval.
- Probability vs. Certainty: Once an interval is computed, the true mean μ either is or isn’t in that interval; the probability interpretation applies only to the process over many samples.

Implications for Statistical Practice

- When variance is known, confidence levels are transparent and directly tied to the critical value $Z_{\alpha/2}$.
- Choosing a higher confidence level (e.g., 99%) results in a wider interval, reflecting increased certainty.
- Conversely, lower confidence levels (e.g., 90%) produce narrower intervals but with less confidence that the interval contains the true mean.

Limitations and Considerations

- Known Variance Assumption: In practice, the population variance is rarely known precisely, which limits the direct application of this theory.
- Normality Assumption: The accuracy of the confidence interval relies on the population being normally distributed, especially for small sample sizes.
- Large Samples: When n is large, the Central Limit Theorem justifies normal approximation even if the population is not strictly normal.

Summary and Final Remarks

- The confidence level for a confidence interval in a normal population with known variance is inherently linked to the critical value $Z_{\alpha/2}$ chosen.
- It represents the long-term proportion of such intervals that will contain the true population mean.
- The process involves selecting a confidence level, calculating the corresponding $Z_{\alpha/2}$, and constructing the interval accordingly.
- Importantly, the confidence level is a property of the construction method, not a probability statement about a specific interval.

In essence, when working with a normally distributed population with known variance, the confidence level is explicitly determined by the critical value from the standard normal distribution, ensuring precise and reliable inference about the population mean.

References for Further Reading

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Final Note:
Understanding the confidence level in this context provides a solid foundation for more complex statistical inference, including cases with unknown variance, non-normal distributions, and Bayesian approaches.

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