In Each Of Problems 1 Through 4, Use The Method Of Variation Of Parameters To Determine The General Solution

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Introduction to the Method of Variation of Parameters

In the study of differential equations, especially linear nonhomogeneous differential equations, finding explicit solutions can be challenging. The Method of Variation of Parameters is a powerful technique that provides a systematic approach to determine the general solution of such equations. Unlike methods tailored specifically for particular types of equations, variation of parameters works broadly, making it a vital tool in both theoretical and applied mathematics.

This method extends the solution of the corresponding homogeneous equation by allowing the constants to vary with the independent variable. Consequently, it constructs a particular solution that accounts for the nonhomogeneous term, ultimately leading to the general solution.

Understanding the Method of Variation of Parameters

The Basic Idea

Suppose you have a second-order linear differential equation:

$$ y'' + p(x) y' + q(x) y = r(x) $$

- The associated homogeneous equation is:

$$ y'' + p(x) y' + q(x) y = 0 $$

- The general solution to the homogeneous equation, denoted as y_h, is typically known or can be found through characteristic equations or other methods.

- The goal is to find a particular solution y_p to the nonhomogeneous equation using variation of parameters.

Key Concept: Instead of treating the constants in y_h as fixed, we allow them to be functions of x. This leads to a solution of the form:

$$ y = u_1(x) y_1(x) + u_2(x) y_2(x) $$

where y_1 and y_2 are fundamental solutions to the homogeneous equation, and u_1, u_2 are functions to be determined.

Derivation of the Method

To find $u_1(x)$ and $u_2(x)$, we impose the following conditions:

$$
\begin{cases}
u_1' y_1 + u_2' y_2 = 0 \\
u_1' y_1' + u_2' y_2' = r(x)
\end{cases}
$$

This system ensures that the derivatives of u_1 and u_2 can be solved explicitly.

Using Cramer's rule, the solutions are:

$$
u_1' = - \frac{y_2 r(x)}{W}
$$
$$
u_2' = \frac{y_1 r(x)}{W}
$$

where W is the Wronskian:

$$
W = y_1 y_2' - y_2 y_1'
$$

Integrating u_1' and u_2' gives the functions u_1 and u_2, leading to the particular solution y_p.

Applying Variation of Parameters to Problems 1 Through 4

The core of this article focuses on four specific problems, each demonstrating the use of variation of parameters to find the general solution. While each problem has unique features, the underlying method remains consistent.

Problem 1: Second-Order Linear Differential Equation with Constant Coefficients

Problem Statement

Solve the differential equation:

$$ y'' + 3 y' + 2 y = e^{x} $$

using the method of variation of parameters.

Step 1: Find the Homogeneous Solution

The homogeneous equation:

$$ y'' + 3 y' + 2 y = 0 $$

Characteristic equation:

$$ r^2 + 3 r + 2 = 0 $$

Factor:

$$ (r + 1)(r + 2) = 0 $$

Solutions:

$$ r = -1, -2 $$

Homogeneous solution:

$$ y_h = C_1 e^{-x} + C_2 e^{-2x} $$

Step 2: Compute the Wronskian

$$
W = y_1 y_2' - y_2 y_1' = \begin{vmatrix}
e^{-x} & e^{-2x} \\
- e^{-x} & -2 e^{-2x}
\end{vmatrix} = e^{-x} (-2 e^{-2x}) - e^{-2x} (- e^{-x}) = -2 e^{-3x} + e^{-3x} = - e^{-3x}
$$

Step 3: Find u_1' and u_2'

Given $r(x) = e^{x}$:

$$
u_1' = - \frac{ y_2 r(x) }{W} = - \frac{ e^{-2x} \times e^{x} }{ - e^{-3x} }

$$
= \frac{ e^{-x} }{ e^{-3x} } = e^{2x}
$$

$$
u_2' = \frac{ y_1 r(x) }{ W } = \frac{ e^{-x} \times e^{x} }{ - e^{-3x} } = \frac{ 1 }{ - e^{-3x} } = - e^{3x}
$$

Step 4: Integrate to find u_1 and u_2

$$
u_1 = \int e^{2x} dx = \frac{1}{2} e^{2x} + K_1
$$

$$
u_2 = - \int e^{3x} dx = - \frac{1}{3} e^{3x} + K_2
$$

Constants of integration can be absorbed into the homogeneous solution; hence, we focus on the particular solution:

$$
y_p = u_1 y_1 + u_2 y_2 = \left(\frac{1}{2} e^{2x} \right) e^{-x} + \left(- \frac{1}{3} e^{3x} \right) e^{-2x}
$$

Simplify:

$$
y_p = \frac{1}{2} e^{x} - \frac{1}{3} e^{x} = \left(\frac{1}{2} - \frac{1}{3} \right) e^{x} = \frac{1}{6} e^{x}
$$

Final answer:

$$
\boxed{
y = C_1 e^{-x} + C_2 e^{-2x} + \frac{1}{6} e^{x}
}
$$

Problem 2: Nonhomogeneous Equation with Variable Coefficients

Problem Statement

Solve:

$$ x^2 y'' - 3 x y' + 4 y = \sin(\ln x), \quad x > 0 $$

using the method of variation of parameters.

Step 1: Homogeneous Equation

Rewrite as:

$$ y'' - \frac{3}{x} y' + \frac{4}{x^2} y = 0 $$

Recognize this as an Euler-type equation. Assume solutions of the form:

$$ y = x^{r} $$

Plug into the homogeneous equation:

$$ r(r - 1) x^{r - 2} - 3 r x^{r - 2} + 4 x^{r - 2} = 0 $$

Divide through by $x^{r - 2}$:

$$ r(r - 1) - 3 r + 4 = 0 $$

Simplify:

$$ r^2 - r - 3 r + 4 = 0 \Rightarrow r^2 - 4 r + 4 = 0 $$

$$ (r - 2)^2 = 0 \Rightarrow r = 2 \text{ (double root)} $$

Homogeneous solution:

$$ y_h = C_1 x^{2} + C_2 x^{2} \ln x $$

Step 2: Fundamental Solutions

$$
y_1 = x^2, \quad y_2 = x^2 \ln x
$$

Calculate Wronskian:

$$
W = y_1 y_2' - y_2 y_1'
$$

Compute derivatives:

$$
y_1' = 2 x, \quad y_2' = 2 x \ln x + x
$$

Calculate W:

$$
W = x^2 (2 x \ln x + x) - x^2 \ln x \times 2 x = x^2 (2 x \ln x + x) - 2 x^3 \ln x
$$

Simplify:

$$

W = 2 x^3 \ln x + x^3 - 2 x^3 \ln x = x^3
\]

Step 3: Find \(u_1' \), \(u_

Frequently Asked Questions

What is the main idea behind the method of variation of parameters when solving differential equations?

The method of variation of parameters involves finding a particular solution to a nonhomogeneous differential equation by allowing the constants in the complementary (homogeneous) solution to vary with the independent variable, thereby constructing a particular solution that satisfies the entire equation.

How do you determine the functions to replace the constants in the complementary solution during the variation of parameters process?

You set up a system of equations using the derivatives of the unknown functions and the Wronskian of the homogeneous solutions, then solve these equations to find the functions that, when substituted back, give the particular solution.

In applying variation of parameters, why is the Wronskian important?

The Wronskian helps determine the linear independence of the solutions to the homogeneous equation and is used in the formulas to compute the functions replacing the constants, ensuring the particular

solution is correctly constructed.

Can the method of variation of parameters be applied to nonlinear differential equations?

No, the method of variation of parameters is specifically designed for linear differential equations. For nonlinear equations, different techniques such as substitution, transformation, or numerical methods are typically used.

How does the process differ when solving second-order versus higher-order differential equations using variation of parameters?

While the core idea remains the same, for higher-order equations, the method involves more linearly independent solutions and a larger system of equations to determine the functions replacing the constants. The calculations become more complex but follow the same principle.

What are common challenges encountered when using variation of parameters, and how can they be addressed?

Challenges include computing the Wronskian, solving the integrals for the functions, and ensuring linear independence. These can be addressed by careful calculation, using substitution or numerical methods for difficult integrals, and verifying the solutions at each step.

Additional Resources

In Each Of Problems 1 Through 4, Use The Method Of Variation Of Parameters To Determine The General Solution

When tackling linear differential equations, especially nonhomogeneous ones, the method of variation of parameters stands out as a powerful and elegant technique. It provides a systematic approach to find particular solutions when the method of undetermined coefficients falls short—particularly for equations with variable coefficients or more complex nonhomogeneous terms. Understanding how to apply this method across different problems is essential for students and professionals dealing with differential equations in mathematics, physics, engineering, and applied sciences. In this comprehensive guide, we will delve into the core principles of the method of variation of parameters, step-by-step procedures, and apply them through multiple illustrative problems.

What Is the Method of Variation of Parameters?

The method of variation of parameters is a technique used to find particular solutions to nonhomogeneous linear differential equations of the form:

$$y'' + p(t) y' + q(t) y = g(t)$$

where $p(t)$, $q(t)$, and $g(t)$ are known functions, and the equation is second order for simplicity.

Core Idea:

- First, solve the corresponding homogeneous equation:

$$ y'' + p(t) y' + q(t) y = 0 $$

to find the complementary (general homogeneous) solution y_c.

- Then, allow the constants in the homogeneous solution to become functions of t instead of constants:

$$ y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t) $$

where $y_1(t)$ and $y_2(t)$ are the fundamental solutions of the homogeneous equation.

- Derive $u_1(t)$ and $u_2(t)$ by solving a system of equations, ensuring the particular solution satisfies the original nonhomogeneous equation.

This approach leverages the structure of the homogeneous solutions but introduces variable coefficients $u_1(t)$ and $u_2(t)$ tailored to the nonhomogeneous term $g(t)$.

Step-by-Step Procedure for Applying Variation of Parameters

Applying the method involves a series of well-defined steps:

1. Solve the Homogeneous Equation
- Find the fundamental solutions $y_1(t)$ and $y_2(t)$.

2. Set Up the Variation of Parameters Form
- Express the particular solution as:

$$ y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t) $$

- To avoid redundancy, impose the auxiliary condition:

$$ u_1'(t) y_1(t) + u_2'(t) y_2(t) = 0 $$

which simplifies calculations.

3. Derive Equations for u_1' and u_2'
- Use the original differential equation to obtain:

$$
\begin{cases}
u_1'(t) y_1(t) + u_2'(t) y_2(t) = 0 \\
u_1'(t) y_1'(t) + u_2'(t) y_2'(t) = g(t)
\end{cases}
$$

- Solve this system to find $u_1'(t)$ and $u_2'(t)$.

4. Integrate to Find $u_1(t)$ and $u_2(t)$
- Integrate the expressions for u_1' and u_2':

$$
u_1(t) = \int u_1'(t) dt, \quad u_2(t) = \int u_2'(t) dt
$$

5. Construct the Particular Solution
- Form $y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$.

6. Write the General Solution
- The general solution to the original nonhomogeneous differential equation is:

$$

$$
y(t) = y_c(t) + y_p(t)
$$

where $y_c(t)$ is the complementary solution and $y_p(t)$ is the particular solution obtained via variation of parameters.

Applying to Problems 1-4: A Guided Approach

Let's now explore how to implement this method across four representative problems. Each problem will involve a different type of nonhomogeneous differential equation, illustrating the versatility of the method.

Problem 1: Second-Order Linear ODE with Constant Coefficients

Equation:

$$ y'' + 3 y' + 2 y = e^{t} $$

Step 1: Homogeneous Solution

Solve:

$$ y'' + 3 y' + 2 y = 0 $$

Characteristic equation:

$$ r^2 + 3 r + 2 = 0 $$

$$ (r + 1)(r + 2) = 0 $$

$$ r = -1, -2 $$

Fundamental solutions:

$$ y_1(t) = e^{-t}, \quad y_2(t) = e^{-2t} $$

Step 2: Set Up Variation of Parameters

Propose:

$$ y_p(t) = u_1(t) e^{-t} + u_2(t) e^{-2t} $$

with the auxiliary condition:

$$ u_1' e^{-t} + u_2' e^{-2t} = 0 $$

Step 3: Derive u_1' and u_2'

Compute:

$$
\begin{cases}
u_1' e^{-t} + u_2' e^{-2t} = 0 \\
u_1' (-e^{-t}) + u_2' (-2 e^{-2t}) = e^{t}
\end{cases}
$$

From the second:

$$ -u_1' e^{-t} - 2 u_2' e^{-2t} = e^{t} $$

Express u_2':

$$ \text{From first: } u_2' = - u_1' e^{t} $$

Plug into second:

$$ - u_1' e^{-t} - 2 (- u_1' e^{t}) e^{-2t} = e^{t} $$

Simplify:

$$ - u_1' e^{-t} + 2 u_1' e^{t} e^{-2t} = e^{t} $$

Since $e^{t} e^{-2t} = e^{-t}$:

$$ - u_1' e^{-t} + 2 u_1' e^{-t} = e^{t} $$

$$ (- u_1' + 2 u_1') e^{-t} = e^{t} $$

$$ u_1' e^{-t} = e^{t} $$

$$ u_1' = e^{t} \cdot e^{t} = e^{2t} $$

Now, $u_2' = - u_1' e^{t} = - e^{2t} e^{t} = - e^{3t}$

Step 4: Integrate

$$
u_1(t) = \int e^{2t} dt = \frac{1}{2} e^{2t} + C_1
$$

$$
u_2(t) = - \int e^{3t} dt = - \frac{1}{3} e^{3t} + C_2
$$

Constants of integration are ignored because they lead to terms absorbed into the homogeneous solution.

Step 5: Particular Solution

$$
y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t) = \frac{1}{2} e^{2t} \cdot e^{-t} - \frac{1}{3} e^{3t} \cdot e^{-2t}
$$

Simplify:

$$
y_p(t) = \frac{1}{2} e^{t} - \frac{1}{3} e^{t}
$$

$$
y_p(t) = \left(\frac{1}{2} - \frac{1}{3} \right) e^{t} = \frac{1}{6} e^{t}
$$

Step 6: General Solution

$$
\boxed{
y(t) = C_1 e^{-t} + C_2 e^{-2t} + \frac{1}{6} e^{t}
}
$$

Problem 2: Variable Coefficients with Non-Constant Forcing

Equation:

$$ t^2 y'' - 3 t y' + 4 y = t^2 $$

Homogeneous Solution:

Rewrite as:

$$ y'' - \frac{3}{t} y' + \frac{4}{t^2} y = 1 $$

The associated homogeneous equation:

$$ t^2 y'' - 3 t y' + 4 y = 0 $$

Dividing through by t^2:

\[y'' - \frac{3}{t} y' + \frac{4

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