Let B B1, B2, B3} Be A Basis Of A Vector Space V. Find T(2b1 +4b2 -3b3) Where T Is The Linear Transformation

Let B = \{B_1, B_2, B_3\} \text{ Be A Basis Of A Vector Space } V. \text{ Find } T(2B_1 + 4B_2 - 3B_3) \text{ Where } T \text{ Is The Linear Transformation}

Introduction to Vector Spaces and Basis

Understanding the problem begins with a solid grasp of the fundamental concepts of vector spaces, bases, and linear transformations. These form the building blocks of linear algebra, a core area of mathematics with applications across engineering, physics, computer science, and beyond.

What Is a Vector Space?

A vector space \(V\) over a field \(F\) (such as the real numbers \(\mathbb{R}\)) is a set equipped with two operations:

- Vector addition: Combining two vectors to produce another vector.
- Scalar multiplication: Multiplying a vector by a scalar from the field.

These operations satisfy specific axioms such as associativity, commutativity of addition, distributivity, and the existence of additive identity and inverses.

What Is a Basis?

A basis $\{B_1, B_2, B_3\}$ of a vector space V is a set of vectors satisfying:

1. Linear independence: No vector in the basis can be written as a linear combination of the others.
2. Spanning set: Every vector in V can be expressed as a linear combination of the basis vectors.

In this context, the basis vectors B_1, B_2, B_3 form a minimal set that can generate the entire vector space V.

Linear Transformations: Definition and Properties

A linear transformation $T: V \to W$ between vector spaces V and W over the same field F is a function satisfying:

- Additivity: $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
- Homogeneity: $T(c\mathbf{v}) = cT(\mathbf{v})$

for all vectors $\mathbf{u}, \mathbf{v} \in V$ and scalars $c \in F$.

Key properties:

- The transformation is completely determined by its action on a basis of V.
- The matrix representation of T relative to bases of V and W provides a concrete method for computation.

Problem Breakdown and Approach

Given:

- Basis $\{B_1, B_2, B_3\}$ of V.
- A linear transformation $T: V \to W$.
- The task: Find $T(2B_1 + 4B_2 - 3B_3)$.

Approach:

1. Express the vector as a linear combination of basis vectors.
2. Use the linearity of T to distribute over the combination.
3. Determine $T(B_1)$, $T(B_2)$, and $T(B_3)$.
4. Compute $T(2B_1 + 4B_2 - 3B_3)$ using the above information.

Step-by-Step Solution

1. Express the vector in terms of basis vectors

The vector in question is:

$$
\mathbf{v} = 2B_1 + 4B_2 - 3B_3
$$

Since the basis vectors are linearly independent and span V, this representation is unique.

2. Use linearity of T

Because T is linear, it satisfies:

$$
T(\mathbf{v}) = T(2B_1 + 4B_2 - 3B_3) = 2T(B_1) + 4T(B_2) - 3T(B_3)
$$

Thus, the problem reduces to knowing $T(B_1)$, $T(B_2)$, and $T(B_3)$.

3. Determine the images of basis vectors under T

Key Point: The images $T(B_1)$, $T(B_2)$, and $T(B_3)$ depend on the linear transformation T. These are often given or can be computed if additional information is provided.

- If the matrix representation of T relative to the bases of V and W is known:
For example, suppose the matrix A of T with respect to bases $\{B_1, B_2, B_3\}$ in V and some basis in W is:

$$
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33} \\
\end{bmatrix}
$$

Then:

$$
T(B_1) = \text{column 1 of } A,\quad T(B_2) = \text{column 2 of } A,\quad T(B_3) = \text{column 3 of } A
$$

- If specific images are given:
For example, if $T(B_1) = W_1$, $T(B_2) = W_2$, $T(B_3) = W_3$, then substitute these directly into the formula.

Example: Computing $T(2B_1 + 4B_2 - 3B_3)$ with Given Data

Suppose, for illustration, the following:

$$
T(B_1) = W_1,\quad T(B_2) = W_2,\quad T(B_3) = W_3
$$

where $W_1, W_2, W_3 \in W$ are known vectors.

Then:

$$
T(2B_1 + 4B_2 - 3B_3) = 2W_1 + 4W_2 - 3W_3
$$

This linear combination provides the image of the vector under T.

Implications and Applications

Understanding how to compute the image of a vector expressed as a linear combination of basis vectors under a linear transformation is fundamental in linear algebra, with numerous applications:

- Coordinate transformations: Changing representations of vectors between different bases.
- Matrix algebra: Computing the action of matrices on vectors.
- Differential equations: Linear operators acting on function spaces.
- Computer graphics: Transforming coordinates in space.
- Machine learning: Feature transformations and linear models.

Further Considerations

Handling Unknowns:

If the explicit form of T or its action on basis vectors isn't provided, the problem cannot be numerically solved. Instead:

- Recognize the process of how the transformation acts.
- Use properties of linearity to simplify computations once data is available.
- Express the answer in terms of $T(B_1)$, $T(B_2)$, and $T(B_3)$.

Change of Basis:

In some cases, it might be necessary to:

- Find the matrix representation of T in different bases.
- Convert vectors from one basis to another.
- Use inverse matrices for basis change if needed.

Summary and Key Takeaways

- A basis $\{B_1, B_2, B_3\}$ provides a unique way to represent any vector in V.
- Linear transformations are completely determined by their action on basis vectors.
- To compute $T(2B_1 + 4B_2 - 3B_3)$, leverage the linearity property.
- The main challenge often lies in knowing $T(B_1)$, $T(B_2)$, and $T(B_3)$, which are typically given or derived from the matrix representation.
- The process exemplifies core concepts in linear algebra, including basis, linearity, and matrix representations.

Conclusion

Understanding how to evaluate the image of a vector under a linear transformation when expressed as a linear combination of basis vectors is a crucial skill in linear algebra. It forms the basis for many advanced topics and practical applications.

In the context of the problem:

$$
\boxed{
T(2B_1 + 4B_2 - 3B_3) = 2T(B_1) + 4T(B_2) - 3T(B_3)
}
$$

The key to solving it is knowing the images of the basis vectors under T. Once these are known, applying the linearity property allows for straightforward computation of the transformation's effect on any linear combination of basis vectors.

Remember:

- Always verify the linear independence of the basis vectors.
- Use the linearity property carefully.
- When in

Frequently Asked Questions

Given a basis B = {B1, B2, B3} of a vector space V, and a linear transformation T, how do you find T(2b1 + 4b2 - 3b3)?

Since T is linear, T(2b1 + 4b2 - 3b3) = 2T(b1) + 4T(b2) - 3T(b3). To evaluate, find T of each basis vector and then combine accordingly.

What information about T is necessary to compute T(2b1 + 4b2 - 3b3)?

You need to know the images of the basis vectors under T, i.e., T(b1), T(b2), and T(b3). Once these are known, you can calculate T of any linear combination using linearity.

If T(b1) = v1, T(b2) = v2, and T(b3) = v3, how can you express T(2b1 + 4b2 - 3b3)?

Using linearity, T(2b1 + 4b2 - 3b3) = 2v1 + 4v2 - 3v3. Substitute the known images to find the specific result.

Why is it important that B = {b1, b2, b3} is a basis when finding T(2b1 + 4b2 - 3b3)?

Because the basis ensures that any vector in V, including 2b1 + 4b2 - 3b3, can be uniquely expressed in terms of B. This allows the use of linearity to compute T easily.

Can T(2b1 + 4b2 - 3b3) be computed without knowing T explicitly? Why or why not?

No, unless T(b1), T(b2), and T(b3) are known. The calculation depends on the images of basis vectors under T; without this information, the exact value cannot be determined.

Additional Resources

Let B = {B□, B□, B□} Be A Basis Of A Vector Space V. Find T(2b□ + 4b□ - 3b□) Where T Is The Linear Transformation

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Introduction

In the realm of linear algebra, understanding how linear transformations operate on vector spaces is fundamental. Given a basis \(B = \{b_1, b_2, b_3\} \) of a vector space \(V \), and a linear transformation $T: V \rightarrow W$ (where W is another vector space), a common problem is to determine the image of a particular vector under T. Specifically, when the vector in question is expressed as a linear combination of basis vectors, it becomes essential to understand the properties of linearity and how they simplify calculations.

This article delves into the process of finding $T(2b_1 + 4b_2 - 3b_3)$ given the basis B and the transformation T. We will explore the theoretical background, the steps involved, and the practical considerations necessary to perform this calculation effectively. Whether you're a student new to linear algebra or a practitioner brushing up on core concepts, this guide aims to clarify the process with clarity and depth.

Understanding the Foundations: Bases, Vectors, and Linear Transformations

What is a Basis in a Vector Space?

A basis in a vector space V is a set of vectors $\{b_1, b_2, ..., b_n\}$ that satisfy two key properties:

1. Linear Independence: No vector in the set can be written as a linear combination of the others.
2. Spanning: Every vector in V can be expressed as a linear combination of the basis vectors.

For the basis $B = \{b_1, b_2, b_3\}$, any vector $v \in V$ can be written uniquely as:

$$v = a_1 b_1 + a_2 b_2 + a_3 b_3$$

where a_1, a_2, a_3 are scalars from the underlying field (commonly real numbers $\mathbb{R}$).

What is a Linear Transformation?

A linear transformation $T: V \rightarrow W$ is a mapping between two vector spaces that preserves vector addition and scalar multiplication:

1. Additivity: $T(u + v) = T(u) + T(v)$ for all $u, v \in V$.
2. Homogeneity: $T(\alpha v) = \alpha T(v)$ for all $v \in V$ and scalars α.

The significance of linear transformations lies in their predictable behavior, which allows complex transformations to be fully characterized by their action on basis vectors.

The Strategy: Calculating $T(2b_1 + 4b_2 - 3b_3)$

Given the linearity of T, the calculation of $T(2b_1 + 4b_2 - 3b_3)$ involves the following steps:

1. Express the vector as a linear combination of basis vectors — which is already provided.
2. Apply the linear transformation to each basis vector — $T(b_1), T(b_2), T(b_3)$.
3. Use linearity to compute the image of the linear combination.

Thus, the key is to know or find out $T(b_1), T(b_2), T(b_3)$. Once these are known, the image of any linear combination can be derived directly.

Step-by-Step Calculation

Step 1: Understanding the action on basis vectors

Suppose the images of the basis vectors under T are known:

\[

$$
T(b_1) = w_1, \quad T(b_2) = w_2, \quad T(b_3) = w_3,
$$

where $w_1, w_2, w_3 \in W$ are known vectors. These could be specified explicitly or given as part of the problem data.

Step 2: Apply linearity

Using the property of linearity:

$$
T(2b_1 + 4b_2 - 3b_3) = 2T(b_1) + 4T(b_2) - 3T(b_3),
$$

which simplifies to:

$$
T(2b_1 + 4b_2 - 3b_3) = 2w_1 + 4w_2 - 3w_3.
$$

This step highlights the core benefit of linearity: the transformation of a linear combination decomposes into the linear combination of the transformations.

Practical Example: Computing $T(2b_1 + 4b_2 - 3b_3)$

To solidify understanding, let's consider a concrete example where the images of the basis vectors are specified.

Suppose:

- $T(b_1) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$,
- $T(b_2) = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$,
- $T(b_3) = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$.

Given this data, the image of the vector $2b_1 + 4b_2 - 3b_3$ under T is:

$$
T(2b_1 + 4b_2 - 3b_3) = 2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} - 3 \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}.
$$

Calculating each term:

- $2 \times \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$,
- $4 \times \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 12 \end{bmatrix}$,
- $-3 \times \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ 0 \\ -3 \end{bmatrix}$.

Adding these vectors yields:

$$
T(2b_1 + 4b_2 - 3b_3) = \begin{bmatrix} 2 + 0 - 12 \\ 0 + 4 + 0 \\ 4 + 12 - 3 \end{bmatrix} = \begin{bmatrix} -10 \\ 4 \\ 13 \end{bmatrix}.
$$

This vector in W is the image of the original linear combination under the transformation T.

Generalization and Additional Considerations

When T is Unknown

In many problems, the images of the basis vectors $T(b_i)$ are not explicitly provided. In such cases, you need to:

- Determine $T(b_i)$ through additional data, such as a matrix representation of T, or
- Express vectors in terms of known images if the transformation is already characterized.

Using Matrix Representation

If the basis vectors are expressed in coordinate form, and T is represented as a matrix $[T]$ with respect to some bases, then:

$$
T(v) = [T] \times v,
$$

where v is the coordinate vector of the original vector in basis B.

This approach simplifies calculations further, especially when dealing with high-dimensional spaces.

Applications and Implications

Understanding how to compute $T(2b_1 + 4b_2 - 3b_3)$ has wide-reaching implications in various fields:

- Computer Graphics: Transforming vectors via linear transformations to rotate, scale, or skew objects.
- Engineering: Analyzing systems where states are transformed linearly.
- Data Science: Applying linear transformations for feature extraction or dimensionality reduction.
- Mathematics: Fundamental in understanding eigenvalues, eigenvectors, and diagonalization.

This fundamental technique also underpins more advanced topics such as eigenvalue problems, matrix decompositions, and the study of linear operators.

Summary and Key Takeaways

- The problem of finding $T(2b_1 + 4b_2 - 3b_3)$ simplifies to applying the linear transformation to each basis vector and then combining the results.
- The linearity property ensures that the transformation of a linear combination equals the corresponding combination of the transformations.
- Accurate calculation depends on knowing $T(b_1), T(b_2), T(b_3)$; once these are known, the process is straightforward.

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