Let G Be The Function Given By G(x)=x2ekx, Where K Is A Constant. For What Value Of K Does G Have A Critical

Let G Be The Function Given By G(x)=x^2 e^{kx}, Where K Is A Constant. For What Value Of K Does G Have A Critical

Understanding the behavior of functions, especially in calculus, often involves analyzing their critical points. Critical points are values of the variable at which the derivative of a function is zero or undefined, and they are crucial in identifying local maxima, minima, or points of inflection. In this article, we will delve into the function $G(x) = x^2 e^{kx}$, where k is a constant, and determine the specific values of k for which G has critical points.

Overview of Critical Points and Their Significance

What Are Critical Points?

Critical points of a function are the points in its domain where the derivative is either zero or undefined. These points can indicate local extrema (maxima or minima) or points of inflection, which are essential in understanding the shape and behavior of the graph of the function.

Mathematically, for a function $G(x)$, the critical points are solutions to:

$$G'(x) = 0 \quad \text{or} \quad G'(x) \text{ is undefined}$$

Why Are Critical Points Important?

Critical points help identify:
- Local maximum and minimum points
- Potential points where the function's increasing/decreasing behavior changes
- Inflection points where the concavity changes

These insights are significant in various fields such as physics, engineering, and economics, where understanding the optimal points or stability regions is essential.

Analyzing the Function $G(x) = x^2 e^{kx}$

Function Composition and Behavior

The given function is a product of a polynomial x^2 and an exponential function e^{kx}. Understanding how these components interact is key to analyzing critical points.

- The polynomial term x^2 is symmetric about the y-axis and has a minimum at $x=0$.
- The exponential term e^{kx} influences the growth or decay depending on the sign of k.

Calculating the Derivative $G'(x)$

Applying the Product Rule

To find the critical points, we start by differentiating $G(x) = x^2 e^{kx}$ with respect to x.

Using the product rule:

$$
G'(x) = \frac{d}{dx}(x^2) \cdot e^{kx} + x^2 \cdot \frac{d}{dx}(e^{kx})
$$

Calculating each term:
- $\frac{d}{dx}(x^2) = 2x$
- $\frac{d}{dx}(e^{kx}) = k e^{kx}$

Putting it together:

$$
G'(x) = 2x e^{kx} + x^2 k e^{kx}
$$

Factor out common terms:

$$
G'(x) = e^{kx} (2x + k x^2)
$$

Finding Critical Points: Solving $G'(x) = 0$

Setting the Derivative Equal to Zero

$$
G'(x) = e^{kx} (2x + k x^2) = 0
$$

Since $e^{kx} \neq 0$ for all real x, the critical points are determined by solving:

$$
2x + k x^2 = 0
$$

Factor the expression:

$$
x (2 + k x) = 0
$$

Hence, the solutions are:

$$
x = 0 \quad \text{or} \quad 2 + k x = 0
$$

From the second equation:

$$
k x = -2 \quad \Rightarrow \quad x = -\frac{2}{k}
$$

Note: The second critical point exists only when $k \neq 0$.

Analyzing the Critical Points Based on K

Case 1: $k \neq 0$

The critical points are:
- $x = 0$
- $x = -\frac{2}{k}$

Implications:
- When $k > 0$, the second critical point is at $x = -\frac{2}{k}$, a negative value.
- When $k < 0$, $x = -\frac{2}{k}$ is positive (since dividing by a negative reverses the sign).

Key observation:
- For $k \neq 0$, there are always two critical points, unless the second solution coincides with the first, which only happens when $k \to \infty$ or 0.

Case 2: $k = 0$

The derivative simplifies to:
$$
G'(x) = e^{0} (2x + 0) = 2x
$$
Set to zero:
$$
2x = 0 \quad \Rightarrow \quad x = 0
$$
Thus, when $k=0$, there is only one critical point at $x=0$.

Conditions for G to Have Critical Points

From the analysis:
- For $k \neq 0$, G always has two critical points: $x=0$ and $x=-\frac{2}{k}$.
- For $k=0$, only one critical point at $x=0$.

Therefore:
- The function G always has critical points for any real k, with the number of critical points depending on whether k is zero or not.

Determining the Nature of Critical Points

To classify the critical points as maxima, minima, or points of inflection, we analyze the second derivative $G''(x)$.

Calculating $G''(x)$

Starting from:
$$
G'(x) = e^{kx} (2x + k x^2)
$$
Apply the product rule again:
$$
G''(x) = \frac{d}{dx} \left[e^{kx} (2x + k x^2) \right]
$$

Derivative:
$$
G''(x) = e^{kx} \cdot k (2x + k x^2) + e^{kx} (2 + 2k x)
$$

$$
G''(x) = e^{kx} \left[k(2x + k x^2) + 2 + 2k x \right]
$$

Simplify inside brackets:

$$
k \cdot 2x + k^2 x^2 + 2 + 2k x = 2k x + k^2 x^2 + 2 + 2k x
$$

$$
= (2k x + 2k x) + k^2 x^2 + 2 = 4k x + k^2 x^2 + 2
$$

Therefore:

$$
G''(x) = e^{kx} (k^2 x^2 + 4k x + 2)
$$

Since $e^{kx} > 0$ for all real x, the concavity depends on the quadratic:

$$
Q(x) = k^2 x^2 + 4k x + 2
$$

Classifying Critical Points Using $G''(x)$

At $x=0$:

$$
Q(0) = 0 + 0 + 2 = 2 > 0
$$

$$
G''(0) > 0
$$

Thus, at $x=0$, the second derivative is positive, indicating a local minimum regardless of k.

At $x= -\frac{2}{k}$:

Calculate $Q(x)$ at $x= -\frac{2}{k}$:

$$
Q\left(-\frac{2}{k}\right) = k^2 \left(-\frac{2}{k}\right)^2 + 4k \left(-\frac{2}{k}\right) + 2
$$

Simplify:

$$
k^2 \cdot \frac{4}{k^2} - 8 + 2 = 4 - 8 + 2 = -2
$$

\]

Since $Q\left(-\frac{2}{k}\right) = -2 < 0$, the second derivative at this critical point is negative, indicating a local maximum at $x = -\frac{2}{k}$ (for $k \neq 0$).

Summary of Critical Point Analysis

Critical Point	Location	Nature	Conditions

Frequently Asked Questions

What is the function G(x) given in the problem?

The function G(x) is given by $G(x) = x^2 e^{kx}$, where K is a constant.

What does it mean for G to have a critical point?

A critical point of G occurs where its derivative G'(x) equals zero or is undefined.

How do we find the derivative G'(x) of the function G(x)?

Using the product rule: $G'(x) = d/dx [x^2] e^{kx} + x^2 d/dx [e^{kx}] = 2x e^{kx} + x^2 (k e^{kx}) = e^{kx}(2x + kx^2)$.

What condition must be satisfied for G to have a critical point?

$G'(x) = e^{kx}(2x + kx^2) = 0$. Since $e^{kx} \neq 0$ for all x, the critical points occur where $2x + kx^2 = 0$.

How do we solve for x in the equation $2x + kx^2 = 0$?

Factor the equation: $x(2 + kx) = 0$, which gives the solutions $x = 0$ or $x = -2/k$, assuming $k \neq 0$.

For which values of K does the function G have critical points?

G has critical points at $x = 0$ for all K, and at $x = -2/k$ when $K \neq 0$. If $K = 0$, then $G(x) = x^2$, which has a critical point at $x = 0$.

What is the significance of the critical points in the context of

the function G?

Critical points indicate potential local maxima, minima, or saddle points, which are important for understanding the function's behavior.

Can G(x) have multiple critical points depending on K?

Yes, when $K \neq 0$, G has two critical points at $x = 0$ and $x = -2/k$, whereas for $K=0$, it only has the critical point at $x=0$.

Additional Resources

Let G Be The Function Given By $G(x) = x^2 e^{kx}$, Where k Is A Constant. For What Value Of K Does G Have A Critical Point?

Introduction

The exploration of critical points in mathematical functions is fundamental in calculus, providing insight into the behavior of functions such as identifying local maxima, minima, or saddle points. In this detailed investigation, we focus on the function:

$$G(x) = x^2 e^{kx}$$

where k is a real constant. Our primary goal is to determine the specific value(s) of k for which $G(x)$ possesses critical points. Critical points occur where the derivative $G'(x)$ is zero or undefined, and analyzing these points reveals the local behavior of the function.

This article aims to provide a comprehensive, step-by-step analysis of the function $G(x)$, including derivative calculations, critical point conditions, and implications of different k values. Through this, we will answer the central question: For what value of k does $G(x)$ have a critical point?

The Significance of Critical Points in Calculus

Understanding critical points is essential in calculus because they mark potential locations of local maxima, minima, or saddle points where the function's increasing or decreasing behavior changes. Determining these points involves:

- Differentiating the function
- Setting the derivative equal to zero
- Solving for the variable x in terms of parameters like k

Once critical points are identified, further analysis, such as second derivative tests, can classify these points.

Deriving the First Derivative $G'(x)$

Given:

$$ G(x) = x^2 e^{kx} $$

we apply the product rule for differentiation:

$$
G'(x) = \frac{d}{dx}(x^2) \cdot e^{kx} + x^2 \cdot \frac{d}{dx}(e^{kx})
$$

Calculating each term:

- $\frac{d}{dx}(x^2) = 2x$
- $\frac{d}{dx}(e^{kx}) = k e^{kx}$

Thus,

$$
G'(x) = 2x e^{kx} + x^2 \cdot k e^{kx}
$$

Factor out the common term e^{kx}:

$$
G'(x) = e^{kx} (2x + k x^2)
$$

Expressed more neatly:

$$
G'(x) = e^{kx} \left(2x + k x^2 \right)
$$

Identifying Critical Points

Critical points occur where $G'(x) = 0$ or is undefined. Since the exponential function e^{kx} is always positive for all real x and real k, it cannot be zero. Therefore, the critical points are determined solely by:

$$
2x + k x^2 = 0
$$

Factor out x:

$$
x (2 + k x) = 0

\]

This yields two possibilities:

1. \(x = 0\)
2. \(2 + k x = 0 \Rightarrow x = -\frac{2}{k}\), provided \(k \neq 0\)

Critical Points in Terms of \(k\)

Case 1: \(k \neq 0\)

- Critical points are at:

\[
x = 0 \quad \text{and} \quad x = -\frac{2}{k}
\]

Case 2: \(k = 0\)

- The derivative simplifies to:

\[
G'(x) = e^{0} (2x + 0) = 2x
\]

- Setting \(G'(x) = 0\):

\[
2x = 0 \Rightarrow x = 0
\]

- So, for \(k=0\), the critical point is at \(x=0\).

Analyzing the Existence of Critical Points

The critical points are explicitly dependent on the value of \(k\). To understand whether these points are meaningful in the context of the function's behavior, we examine:

- The nature of these points
- Their implications for the function's local maxima or minima
- Whether the critical points exist for all real \(k\)

Deep Dive: Behavior of \(G(x)\) for Different \(k\) Values

When \(k = 0\)

- The function reduces to:

$$
G(x) = x^2
$$

- It has a global minimum at $x=0$, where $G(0) = 0$

- The critical point at $x=0$ corresponds to this minimum

When $k > 0$

- Critical points at $x=0$ and $x = -\frac{2}{k}$

- Since $k > 0$, $-\frac{2}{k} < 0$, indicating two critical points on the real line:

- $x=0$, the origin
- A negative critical point at $x = -\frac{2}{k}$

- The behavior of the function around these points can be analyzed to determine whether they are maxima or minima

When $k < 0$

- Critical points at $x=0$ and $x = -\frac{2}{k}$

- Because $k<0$, $-\frac{2}{k}$ is positive, leading to critical points at:

- $x=0$
- $x = -\frac{2}{k} > 0$

- Again, the nature of these critical points depends on the second derivative analysis

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Second Derivative and Critical Point Classification

To classify the critical points, compute the second derivative:

$$
G''(x) = \frac{d}{dx} G'(x) = \frac{d}{dx} \left[e^{kx} (2x + k x^2) \right]
$$

Applying the product rule again:

$$
G''(x) = e^{kx} \left(k (2x + k x^2) + \frac{d}{dx}(2x + k x^2) \right)
$$

Calculating:

$$
\frac{d}{dx}(2x + k x^2) = 2 + 2k x
$$

Thus:

$$
G''(x) = e^{kx} \left[k (2x + k x^2) + 2 + 2k x \right]
$$

Simplify the expression inside brackets:

$$
k(2x + k x^2) + 2 + 2k x = 2k x + k^2 x^2 + 2 + 2k x
$$

Combine like terms:

$$
(2k x + 2k x) + k^2 x^2 + 2 = 4k x + k^2 x^2 + 2
$$

Therefore,

$$
G''(x) = e^{kx} \left(k^2 x^2 + 4k x + 2 \right)
$$

Since $e^{kx} > 0$ for all real x, the sign of $G''(x)$ depends on the quadratic:

$$
Q(x) = k^2 x^2 + 4k x + 2
$$

Conditions for the Nature of Critical Points

- If $Q(x) > 0$ at a critical point, the point is a local minimum.
- If $Q(x) < 0$ at a critical point, the point is a local maximum.

To identify the nature of the critical points at $x=0$ and $x = -2/k$, evaluate $Q(x)$ at these points.

Evaluating $Q(x)$ at Critical Points

At $x=0$:

$$
Q(0) = k^2 \cdot 0 + 4k \cdot 0 + 2 = 2
$$

\]

Since \(Q(0) = 2 > 0\), the critical point at \(x=0\) is always a local minimum, regardless of \(k\).

At \(x = -2/k\):

Plug into \(Q(x)\):

\[
Q\left(-\frac{2}{k}\right) = k^2 \left(\frac{4}{k^2} \right) + 4k \left(-\frac{2}{k} \right) + 2
\]

Simplify:

\[
= 4 + (-8) + 2 = (4 - 8 + 2) = -2
\]

Thus,

\[
Q \left(-\frac{2}{k} \right) = -2 < 0
\]

This indicates that at \(x = -2/k\), the critical point is a local maximum, regardless of the value of \(k\) (excluding the trivial case \(k=0\), where the critical point coincides with the minimum at zero).

Summary of Critical Point Analysis

- Critical point at \(x=

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