Tutorial Exercise Use Newton's Method To Find The Absolute Maximum Value Of The Function F(x) = 14x Cos(x),

Tutorial Exercise Use Newton's Method To Find The Absolute Maximum Value Of The Function F(x) = 14x Cos(x), in this comprehensive guide, we will explore how to apply Newton's Method to locate the absolute maximum of the function $F(x) = 14x \cos(x)$. This process involves understanding the function's behavior, calculating derivatives, and iteratively refining guesses to approach the maximum point with precision. Whether you're a calculus student or a professional seeking to reinforce your numerical methods skills, this tutorial provides step-by-step instructions, detailed explanations, and practical insights.

Understanding the Function $F(x) = 14x \cos(x)$

Before delving into Newton's Method, it's essential to analyze the function's properties, domain, and differentiability to identify potential maximum points.

Function Overview

- Type of Function: $F(x)$ combines polynomial and trigonometric components—a linear term multiplied by a cosine function, resulting in a oscillatory behavior with amplitude increasing proportionally to x.
- Domain: $x \in (-\infty, \infty)$. However, in practical applications or for the purpose of maximum search, we often restrict the domain to relevant intervals where the function attains its global maximum.

Behavior Analysis

- As $x \to \infty$, $F(x)$ oscillates with increasing amplitude, but the cosine term oscillates between -1 and 1, so the maximum value at large x is unbounded in theory.
- To find the absolute maximum, we analyze critical points where the derivative $F'(x)$ equals zero, as maxima occur at critical points where the function changes from increasing to decreasing.

Calculating the Derivatives of $F(x)$

Newton's Method requires the first and second derivatives of the function to iteratively approximate the critical points.

First Derivative $F'(x)$

Given:

$$
F(x) = 14x \cos(x)
$$

Apply the product rule:

$$
F'(x) = 14 \left(\frac{d}{dx}(x) \cdot \cos(x) + x \cdot \frac{d}{dx}(\cos(x)) \right)
$$

$$
F'(x) = 14 \left(1 \cdot \cos(x) + x \cdot (-\sin(x)) \right)
$$

$$
F'(x) = 14 \left(\cos(x) - x \sin(x) \right)
$$

Second Derivative $F''(x)$

Differentiate $F'(x)$:

$$
F''(x) = 14 \left(-\sin(x) - \sin(x) - x \cos(x) \right)
$$

$$
F''(x) = 14 \left(-2 \sin(x) - x \cos(x) \right)
$$

These derivatives are crucial for Newton's Method, which uses the formula:

$$
x_{n+1} = x_n - \frac{F'(x_n)}{F''(x_n)}
$$

Identifying Critical Points

Critical points occur where $F'(x) = 0$:

$$
\cos(x) - x \sin(x) = 0
$$

or equivalently:

$$
\cos(x) = x \sin(x)
$$

This transcendental equation doesn't have a closed-form solution, so numerical methods like Newton's Method are used to find approximate solutions.

Initial Guesses

- Since $\cos(x)$ and $\sin(x)$ are periodic, initial guesses can be chosen based on the approximate locations of maxima.
- For $x > 0$, the first positive critical point is near $x = 0$. Additional critical points can be estimated around multiples of π, considering the behavior of sine and cosine functions.

Applying Newton's Method to Find Critical Points

Let's now detail the iterative process to approximate the critical point(s) where $F'(x) = 0$.

Step-by-Step Process

1. Choose an initial guess x_0 based on the analysis above.
2. Compute $F'(x_0)$ and $F''(x_0)$.
3. Update the estimate:
$$
x_{n+1} = x_n - \frac{F'(x_n)}{F''(x_n)}
$$
4. Repeat the process until the change $|x_{n+1} - x_n|$ is below a specified tolerance (e.g., 10^{-6}).

Example Calculation

Suppose we take an initial guess $x_0 = 1$.

- Compute:
$$
F'(1) = 14 (\cos(1) - 1 \times \sin(1))
$$
- Compute:
$$
F''(1) = 14 (-2 \sin(1) - 1 \times \cos(1))
$$

Calculate numerical values:
- $\cos(1) \approx 0.5403$
- $\sin(1) \approx 0.8415$

Then:
$$
F'(1) \approx 14 (0.5403 - 0.8415) = 14 (-0.3012) \approx -4.217
$$
$$
F''(1) \approx 14 (-2 \times 0.8415 - 1 \times 0.5403) = 14 (-1.683 - 0.5403) = 14 (-2.2233)

$$
\approx -31.106
$$

Update:

$$
x_1 = 1 - \frac{-4.217}{-31.106} \approx 1 - 0.1357 \approx 0.8643
$$

Repeat the process with $x_1 = 0.8643$, iterating until convergence.

Locating the Absolute Maximum

Once the critical points are approximated, evaluate $F(x)$ at these points to identify which yields the maximum value.

Sample Critical Points and Corresponding Function Values

- For $x \approx 0$:

$$
F(0) = 14 \times 0 \times \cos(0) = 0
$$

- For the critical point near $x \approx 1.4$ (from iterative approximation):

$$
F(1.4) \approx 14 \times 1.4 \times \cos(1.4)
$$

Calculate:

$$
\cos(1.4) \approx 0.1736
$$

$$
F(1.4) \approx 14 \times 1.4 \times 0.1736 \approx 14 \times 0.243 \approx 3.4
$$

Similarly, check at other critical points (e.g., near $x \approx 4.4$, etc.) to find the global maximum in the domain.

Interpreting Results and Confirming Maxima

To confirm whether a critical point corresponds to a maximum:

- Use the second derivative test:
- If $F''(x) < 0$, the point is a local maximum.
- If $F''(x) > 0$, it's a local minimum.
- Evaluate $F(x)$ at the critical points, and compare to find the absolute maximum.

In practice:
- For the function $F(x) = 14x \cos(x)$, due to oscillatory behavior, the absolute maximum occurs at a critical point where the function attains the highest value among all critical points.

Practical Considerations in Using Newton's Method

When applying Newton's Method, keep in mind:

- Initial Guess Sensitivity: Different starting points may converge to different critical points.
- Derivative Computations: Ensure derivatives are correctly calculated to avoid errors.
- Convergence: The method converges quadratically near the root but may fail if $F''(x)$ is zero or very small.
- Alternative Methods: For functions with multiple maxima, consider combining Newton's Method with other techniques like bisection or secant methods to locate all critical points.

Conclusion and Summary

Applying Newton's Method to find the absolute maximum of $F(x) = 14x \cos(x)$ involves:

- Deriving the first and second derivatives.
- Solving $F'(x) = 0$ numerically via iterative methods.
- Evaluating the function at critical points to identify the global maximum.
- Verifying the nature of the critical points through the second derivative test.

This procedure enables precise approximation of the maximum value, essential in optimization problems involving oscillatory functions with polynomial factors. Remember, understanding the behavior of the function guides effective initial guesses and improves convergence efficiency.

Additional Tips for Optimization Problems

- Visualize the function graphically to estimate critical points.
- Use software

Frequently Asked Questions

What is the goal when using Newton's Method to find the absolute maximum of the function F(x) = 14x

Cos(x)?

The goal is to find the value of x where F(x) attains its highest point (maximum), by iteratively applying Newton's Method to solve for critical points where the first derivative equals zero.

How do you determine the critical points of F(x) = 14x Cos(x)?

Critical points occur where the first derivative F'(x) = 0. First, compute F'(x) and then solve for x to find potential maxima or minima.

What are the steps to apply Newton's Method to find the maximum of F(x)?

First, find F'(x) and F''(x). Set up the equation F'(x) = 0 and then use Newton's iteration formula $x_{n+1} = x_n - F'(x_n)/F''(x_n)$ to converge to a critical point that is a maximum.

How do you determine if a critical point found via Newton's Method is a maximum?

Check the second derivative F''(x) at the critical point. If F''(x) < 0, the point is a local maximum. For the absolute maximum, compare the function values at all critical points.

What is the derivative F'(x) of the function F(x) = 14x Cos(x)?

Using the product rule, F'(x) = 14 Cos(x) - 14x Sin(x).

What is the second derivative F''(x) of F(x) = 14x Cos(x)?

Differentiating F'(x), we get F''(x) = -28 Sin(x) - 14 Sin(x) - 14x Cos(x) = -42 Sin(x) - 14x Cos(x).

How do you choose an initial guess for Newton's Method in this problem?

Select an initial x value close to where you expect the maximum, such as x = 0 or based on the behavior of the function, to ensure convergence towards the maximum point.

Can Newton's Method be used directly to find the maximum without solving F'(x) = 0?

No, Newton's Method is typically used to find zeros of derivatives. To find the maximum, you first find critical points by solving F'(x) = 0 using Newton's Method, then determine which are maxima.

Is it necessary to check the endpoints when finding the absolute maximum of F(x)?

Yes, if the domain is restricted, it is important to compare the function values at critical points and endpoints to determine the absolute maximum.

What is the significance of using Newton's Method in this context?

Newton's Method provides an efficient numerical approach to accurately locate the critical points where the maximum occurs, especially when an analytical solution is complex or difficult.

Additional Resources

Mastering Newton's Method to Find the Absolute Maximum of F(x) = 14x Cos(x)

Introduction

Calculus offers a diverse toolkit for analyzing functions, especially when it comes to identifying their maximum and minimum values. Among these techniques, Newton's method stands out as a powerful iterative approach to find roots of equations, which can be instrumental in locating critical points where the function reaches its extrema. In this comprehensive guide, we will explore how to apply Newton's method to determine the absolute maximum of the function:

$$
F(x) = 14x \cos(x)
$$

We will delve into each step meticulously, providing insights into the calculus involved, the iterative process, and the interpretation of results. This tutorial aims to equip you with both conceptual understanding and practical skills to tackle similar optimization problems using Newton's method.

Understanding the Function $F(x) = 14x \cos(x)$

Before deploying Newton's method, it is essential to comprehend the behavior of the function:

- Nature of the Function: The function multiplies a linear term $14x$ with a trigonometric term $\cos(x)$. Such functions are oscillatory with increasing amplitude as x grows, but the linear component influences the overall trend.

- Domain: Typically, for maximum/minimum analysis, the domain can be extended over all real numbers, but practically, the function's maxima are often within certain intervals because $\cos(x)$ oscillates between -1 and 1.

- Expected Behavior: The maxima of $F(x)$ will occur at points where the first derivative $F'(x)$ equals zero, indicating critical points, which could be maxima, minima, or saddle points.

Step 1: Find Critical Points by Derivative Analysis

To locate maxima, we need the critical points where the derivative $F'(x)$ equals zero.

Calculating $F'(x)$:

Given:

$$
F(x) = 14x \cos(x)
$$

Use the product rule:

$$
F'(x) = 14 \left(\frac{d}{dx} [x \cos(x)] \right) = 14 \left(\cos(x) + x (-\sin(x)) \right)
$$

Simplify:

$$
F'(x) = 14 \left(\cos(x) - x \sin(x) \right)
$$

Set $F'(x) = 0$ to find critical points:

$$
14 (\cos(x) - x \sin(x)) = 0
$$

Dividing both sides by 14:

$$
\cos(x) - x \sin(x) = 0
$$

which simplifies to:

$$

\boxed{\cos(x) = x \sin(x)}
\]

This is a transcendental equation. Solving it explicitly for \(x\) is not feasible analytically, hence the need for numerical methods like Newton's method.

Step 2: Applying Newton's Method to Find Critical Points

Newton's method is an iterative root-finding algorithm that refines guesses to solutions of equations \(f(x) = 0\).

Procedure:

- For a function \(f(x)\), starting from an initial guess \(x_0\), the iterative formula is:

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]

- The process repeats until convergence, i.e., until \(|x_{n+1} - x_n|\) is below a specified tolerance.

Defining \(f(x)\):

In our case:

\[
f(x) = \cos(x) - x \sin(x)
\]

Calculating \(f'(x)\):

Differentiate \(f(x)\):

\[
f'(x) = -\sin(x) - \left(\sin(x) + x \cos(x) \right) = -\sin(x) - \sin(x) - x \cos(x) = -2 \sin(x) - x \cos(x)
\]

Summary:

\[
\boxed{
\begin{aligned}
f(x) &= \cos(x) - x \sin(x) \\
f'(x) &= -2 \sin(x) - x \cos(x)
\end{aligned}
}
\]

Step 3: Implementing Newton's Method

Initial guesses:

- Since $\cos(x)$ and $\sin(x)$ are periodic, critical points often occur near multiples of $\pi/2$.

- For positive x, start with guesses around $x = 0, \pi/2, \pi, 3\pi/2, 2\pi$, etc.

Example:

- Let's take $x_0 = 1$ as an initial guess.

Iterative process:

Calculate:

$$
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
$$

- Iteration 1:

- $f(1) = \cos(1) - 1 \times \sin(1) \approx 0.5403 - 0.8415 = -0.3012$

- $f'(1) = -2 \times \sin(1) - 1 \times \cos(1) \approx -2 \times 0.8415 - 0.5403 = -1.6830 - 0.5403 = -2.2233$

- $x_1 = 1 - \frac{-0.3012}{-2.2233} \approx 1 - 0.1354 = 0.8646$

- Iteration 2:

- $f(0.8646) \approx \cos(0.8646) - 0.8646 \times \sin(0.8646)$

$$
\cos(0.8646) \approx 0.6489,\quad \sin(0.8646) \approx 0.7600
$$

$$
f(0.8646) \approx 0.6489 - 0.8646 \times 0.7600 \approx 0.6489 - 0.6572 = -0.0083
$$

- $f'(0.8646) \approx -2 \times 0.7600 - 0.8646 \times 0.6489 \approx -1.5200 - 0.5610 = -2.0810$

- $x_2 = 0.8646 - \frac{-0.0083}{-2.0810} \approx 0.8646 - 0.0040 = 0.8606$

- Iteration 3:

- Repeating similar calculations, the value converges around $x \approx 0.86$.

Result:

- The critical point near this initial guess is approximately at $x \approx 0.86$.

- Similar procedures can be applied for other initial guesses to find additional critical points, especially negative or larger positive x.

Step 4: Determining whether Critical Points are Maxima or Minima

Having identified critical points, the next step is to classify them:

- Calculate the second derivative $F''(x)$:

$$
F''(x) = \frac{d}{dx} F'(x) = \frac{d}{dx} \left(14(\cos(x) - x \sin(x)) \right) = 14 (-\sin(x) - \sin(x) - x \cos(x)) = 14 (-2 \sin(x) - x \cos(x))
$$

- Alternatively, evaluate the second derivative at the critical point:

$$
F''(x) = 14 \times f'(x) = 14 \times (-2 \sin(x) - x \cos(x))
$$

- If $F''(x) < 0$: the critical point is a local maximum.

- If $F''(x) > 0$: the critical point is a local minimum.

Example:

At $x \approx 0.86$:

- $\sin(0.86) \approx 0.76$, $\cos(0.86) \approx 0.65$

- $F''(0.86) = 14 (-2 \times 0.76 - 0.86 \times 0.65) = 14 (-1.52 - 0.56) = 14 \times -2.08 \approx -29.12$

- Since $F''(0.86) < 0$, this critical point corresponds to a local maximum.

Step 5: Finding the Absolute Maximum

While critical points indicate local extrema, to identify the absolute maximum, we need to compare the function's values at these points and consider the endpoints or behavior at infinity.

Calculate $F(x)$ at critical points:

$$
F(x) = 14x \cos(x)
$$

- At $x \approx 0.86$:

$$
F(0.86) \approx 14 \times 0.86 \times
$$

Tutorial Exercise Use Newton S Method To Find The Absolute Maximum Value Of The Function F X 14x Cos X

Tutorial Exercise Use Newton S Method To Find The Absolute Maximum Value Of The Function F X 14x Cos X

Related Articles

- Refer To The Exhibit. A Network Administrator Is Configuring A Router For DHCPv6 Operation. Which Conclusion
- What Would This Situation Look Like From My Supervisors Standpoint? This Is A Question Of ________________a.
- In Pea Plants, Purple Flower Color, C, Is Dominant To White Flower Color, C. The Table Shows The Frequencies

Back to Home