Consider A Rectangular Potential Barrier: 0 Otherwise. With Vo >0 And A > 0. Show That The Transmission

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Introduction to Quantum Tunneling and Potential Barriers

Quantum mechanics introduces phenomena that defy classical intuition, one of which is the concept of quantum tunneling. In classical physics, a particle with energy less than a potential barrier is reflected completely; it cannot penetrate or pass through the barrier. However, quantum particles are described by wave functions that can extend into classically forbidden regions, making it possible, with some probability, to tunnel through barriers even when their energy is less than the barrier height.

This phenomenon has profound implications in various fields, including nuclear physics, semiconductor physics, and quantum computing. To analyze tunneling quantitatively, physicists typically model barriers as potential energy functions and solve the Schrödinger equation to obtain transmission and reflection coefficients.

Modeling the Rectangular Potential Barrier

The Potential Profile

The potential barrier under consideration is modeled as a rectangular barrier:

$$
V(x) =
\begin{cases}
V_0, & \text{for } 0 \leq x \leq A \\
0, & \text{otherwise}
\end{cases}
$$

where:

- $V_0 > 0$ is the height of the barrier.
- $A > 0$ is the width of the barrier.

This simple model captures the essence of tunneling phenomena and allows for analytical solutions to the Schrödinger equation.

Wave Function Regions

We consider a particle with energy E approaching the barrier from the left. The wave functions in different regions are:

- Region I ($x < 0$): Incident wave and reflected wave
- Region II ($0 \leq x \leq A$): Wave within the barrier
- Region III ($x > A$): Transmitted wave

Solving the Schrödinger Equation

The Schrödinger Equation

For a particle of mass m, the time-independent Schrödinger equation is:

$$
-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + V(x)\psi(x) = E \psi(x)
$$

where $\hbar$ is the reduced Planck's constant.

Wave Function Solutions in Different Regions

Region I ($x < 0$):

$$
\psi_I(x) = Ae^{ikx} + Be^{-ikx}
$$

where:

$$
k = \frac{\sqrt{2mE}}{\hbar}
$$

- Ae^{ikx}: incident wave
- Be^{-ikx}: reflected wave

Region II ($0 \leq x \leq A$):

$$
\psi_{II}(x) = Ce^{iqx} + De^{-iqx}
$$

where:

$$
q = \frac{\sqrt{2m(V_0 - E)}}{\hbar}
$$

Note: Since $E < V_0$, q is real and positive, indicating an exponentially decaying wave inside the barrier.

Region III ($x > A$):

$$
\psi_{III}(x) = Fe^{ikx}
$$

representing the transmitted wave traveling to the right.

Applying Boundary Conditions

To determine the transmission coefficient, we need to match the wave functions and their derivatives at the boundaries $x=0$ and $x=A$:

- Continuity of ψ:
$$
\psi_I(0) = \psi_{II}(0), \quad \psi_{II}(A) = \psi_{III}(A)
$$
- Continuity of $d\psi/dx$:
$$
\frac{d\psi_I}{dx}\bigg|_{x=0} = \frac{d\psi_{II}}{dx}\bigg|_{x=0}, \quad
\frac{d\psi_{II}}{dx}\bigg|_{x=A} = \frac{d\psi_{III}}{dx}\bigg|_{x=A}
$$

By applying these boundary conditions, we derive relationships between the coefficients A, B, C, D, F.

Deriving the Transmission Coefficient

Expression for Transmission Coefficient T

The key quantity of interest is the transmission coefficient T, defined as the ratio of transmitted flux to incident flux:

$$
T = \frac{\text{transmitted current density}}{\text{incident current density}} = \left| \frac{F}{A} \right|^2 \frac{k}{k} = \left| \frac{F}{A} \right|^2
$$

The detailed derivation yields the well-known expression:

$$
T = \frac{1}{1 + \frac{V_0^2 \sinh^2(qA)}{4E(V_0 - E)}}
$$

which simplifies to:

$$
T = \frac{1}{1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2(qA)}
$$

This formula indicates that, even for $E < V_0$, the transmission probability is non-zero, illustrating quantum tunneling.

Analysis of the Transmission Coefficient

Behavior for Different Regimes

- Thin and Low Barriers ($A \to 0$, $V_0 \to 0$):

$$
T \to 1
$$

Almost complete transmission, as expected classically.

- High and Wide Barriers ($V_0 \gg E$, $A \to \infty$):

$$
T \to 0
$$

Very low probability of tunneling, consistent with classical intuition.

- Intermediate Regimes:

The tunneling probability depends exponentially on the width A and the barrier height V_0, governed mainly by the hyperbolic sine term $\sinh^2(qA)$.

Approximate Expression for Large qA

When $qA \gg 1$, $\sinh(qA) \approx \frac{1}{2} e^{qA}$, leading to:

$$
T \approx \frac{16E(V_0 - E)}{V_0^2} e^{-2qA}
$$

This exponential decay emphasizes the sensitivity of tunneling probability to barrier parameters.

Physical Interpretation and Significance

The derivation of the transmission coefficient from the rectangular barrier model reveals that quantum particles can penetrate barriers with finite probability, a phenomenon absent in classical physics. The probability diminishes exponentially with increasing barrier width and height but remains nonzero, enabling phenomena such as nuclear fusion in stars and operation of tunnel diodes.

Furthermore, understanding the dependence of T on parameters V_0, A, and E enables engineers and physicists to design quantum devices with desired tunneling characteristics.

Conclusion

By modeling the potential barrier as a rectangular step and solving the Schrödinger equation with appropriate boundary conditions, we derive an explicit expression for the transmission coefficient T. This derivation confirms that even when $E < V_0$, quantum particles possess a finite probability of tunneling through the barrier. The key factors influencing tunneling probability include the barrier height V_0, width A, and the particle's energy E.

The analysis underscores the fundamental departure of quantum mechanics from classical intuition and provides the mathematical foundation for numerous technological innovations such as tunnel diodes, scanning tunneling microscopes, and quantum computing components.

Frequently Asked Questions

What is the general setup for analyzing quantum tunneling through a rectangular potential barrier?

The setup involves a particle approaching a region of space where the potential energy is a constant $V_o > 0$ over a finite width A, and zero elsewhere. The goal is to analyze the transmission and reflection probabilities of the particle tunneling through this barrier using Schrödinger's equation.

How do you define the potential energy function for a rectangular barrier?

The potential energy function $V(x)$ is defined as $V(x) = 0$ for $x < 0$ and $x > A$, and $V(x) = V_o > 0$ for $0 \leq x \leq A$.

What are the key wavefunctions in regions outside and inside the barrier?

Outside the barrier ($x < 0$ and $x > A$), the wavefunction is a superposition of incident and reflected waves, typically written as $\psi = Ae^{ikx} + Be^{-ikx}$. Inside the barrier ($0 \leq x \leq A$), the wavefunction is a combination of exponential decay and growth, expressed as $\psi = Ce^{\kappa x} + De^{-\kappa x}$, where κ relates to the potential height V_o.

How is the wavevector k related to the particle's energy E, and what is κ inside the barrier?

Outside the barrier, $k = \sqrt{2mE}/\hbar$, representing the wavevector for a free particle. Inside the barrier, $\kappa = \sqrt{2m(V_o - E)}/\hbar$, which governs the exponential decay or growth of the wavefunction within the barrier when $E < V_o$.

How do you derive the transmission coefficient T for a rectangular potential barrier?

T is derived by applying boundary conditions at x=0 and x=A to match wavefunctions and their derivatives, resulting in an expression involving k, κ, and A. The standard formula is $T = [1 + (V_o^2 \sinh^2(\kappa A))/(4E(V_o - E))]^{-1}$, which quantifies the probability of tunneling through the barrier.

What is the physical significance of the transmission

probability T in quantum tunneling?

T represents the likelihood that a particle incident on the barrier will pass through it despite not having enough classical energy to overcome the potential. It highlights the quantum phenomenon where particles can tunnel through classically forbidden regions, with T depending on barrier parameters and particle energy.

How does increasing the barrier width A affect the transmission probability T?

Increasing A generally decreases T exponentially, as the wavefunction decays more within a wider barrier. This results in a lower probability of tunneling, illustrating the sensitivity of quantum tunneling to barrier thickness.

Can the transmission probability T be greater than 1? Why or why not?

No, T cannot be greater than 1 because it represents a probability, which must be between 0 and 1. Quantum tunneling probabilities are always less than or equal to one, reflecting the maximum likelihood of transmission through the barrier.

Additional Resources

Quantum Tunneling Through a Rectangular Potential Barrier: An In-Depth Analysis

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Introduction

Quantum mechanics introduces phenomena that defy classical intuition, among which quantum tunneling stands out as particularly fascinating. Unlike classical particles, quantum particles possess a non-zero probability to penetrate and pass through potential barriers—even when their energy is less than the height of the barrier. This behavior is fundamental to many physical systems, from nuclear fusion in stars to modern semiconductor devices.

In this detailed discussion, we focus on a canonical problem: a rectangular potential barrier with height $V_0 > 0$ and width $A > 0$. We aim to demonstrate that, despite the particle’s energy E being less than V_0, there exists a finite transmission probability—that is, the particle can "tunnel" through the barrier.

Setting Up the Problem

The Potential Profile

The potential $V(x)$ is modeled as a rectangular barrier:

$$
V(x) =
\begin{cases}
0, & x < 0 \\
V_0, & 0 \leq x \leq A \\
0, & x > A
\end{cases}
$$

where:
- $V_0 > 0$ (the barrier height),
- $A > 0$ (the barrier width).

This simple potential profile provides a tractable model that captures the essence of tunneling phenomena.

Physical Assumptions

- The particle is described by a wave function $\psi(x)$ satisfying the time-independent Schrödinger equation.
- The particle incident from the left (region I, $x < 0$) with energy E.
- The energy E is less than V_0, i.e., $E < V_0$.

The Schrödinger Equation and Its Solutions

The Schrödinger equation for a particle of mass m is:

$$
-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + V(x) \psi(x) = E \psi(x)
$$

The solutions differ across the three regions:

Region I ($x < 0$): Free particle

$$
V(x) = 0
$$

The general solution:

$$
\psi_I(x) = Ae^{ikx} + Be^{-ikx}
$$

where:

$$
k = \frac{\sqrt{2mE}}{\hbar}
$$

$$

- Ae^{ikx}: incident wave moving to the right.
- Be^{-ikx}: reflected wave moving to the left.

For analysis, we typically normalize the incident wave amplitude to 1 ($A=1$), so:

$$
\psi_I(x) = e^{ikx} + R e^{-ikx}
$$

where $R = \frac{B}{A}$ is the reflection amplitude.

Region II ($0 \leq x \leq A$): Barrier region

$$
V(x) = V_0
$$

Since $E < V_0$, the solutions are exponentially decaying and growing:

$$
\psi_{II}(x) = C e^{\kappa x} + D e^{-\kappa x}
$$

with:

$$
\kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}
$$

This exponential behavior reflects the classically forbidden nature within the barrier.

Region III ($x > A$): Free particle

$$
V(x) = 0
$$

The transmitted wave:

$$
\psi_{III}(x) = T e^{ikx}
$$

where T is the transmission amplitude.

Boundary Conditions and Continuity Equations

Quantum wave functions and their derivatives must be continuous at the boundaries \(x=0 \) and \(x=A \). Specifically:

- At \(x=0 \):

\[
\psi_I(0) = \psi_{II}(0) \Rightarrow 1 + R = C + D
\]
\[
\left. \frac{d\psi_I}{dx} \right|_{0} = \left. \frac{d\psi_{II}}{dx} \right|_{0} \Rightarrow ik (1 - R) = \kappa (C - D)
\]

- At \(x=A \):

\[
\psi_{II}(A) = \psi_{III}(A) \Rightarrow C e^{\kappa A} + D e^{-\kappa A} = T e^{ikA}
\]
\[
\left. \frac{d\psi_{II}}{dx} \right|_{A} = \left. \frac{d\psi_{III}}{dx} \right|_{A} \Rightarrow \kappa (C e^{\kappa A} - D e^{-\kappa A}) = ik T e^{ikA}
\]

These boundary conditions form a system of equations to solve for \(R \) and \(T \).

Derivation of the Transmission Coefficient \(T \)

The algebra involved in solving for \(T \) is extensive but well-documented. The key outcome is the expression for the transmission probability \(T \), given by:

\[
T = \left| \frac{T}{A} \right|^2
\]

which, after solving the boundary conditions, simplifies to:

\[
T = \frac{1}{1 + \frac{V_0^2 \sinh^2 (\kappa A)}{4 E (V_0 - E)}}
\]

or equivalently,

\[
T = \frac{1}{1 + \frac{V_0^2}{4 E (V_0 - E)} \sinh^2 (\kappa A)}
\]

where:

\[

$$
\kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}
$$

This formula encapsulates how the probability of tunneling depends on the barrier's height, width, and the particle's energy.

Physical Interpretation of the Transmission Probability

- Exponential decay within the barrier: The $\sinh^2(\kappa A)$ term indicates that the wave function decays exponentially in the barrier region, leading to a tunneling probability that diminishes with increasing A and κ.

- Barrier parameters: As V_0 increases or A widens, κ increases, reducing T. Conversely, increasing the particle energy E (but still less than V_0) enhances T.

- Finite tunneling probability: Despite $E < V_0$, T is strictly greater than zero, demonstrating the quantum mechanical allowance for tunneling—a phenomenon impossible in classical physics.

Special Cases and Limits

Thin and Low Barriers

- When $A \to 0$, $\sinh (\kappa A) \to 0$, and $T \to 1$, meaning the particle transmits almost perfectly through an infinitesimally thin barrier.

- When $V_0 \to 0$, the barrier disappears, and $T \to 1$, as expected.

Thick and High Barriers

- As $A \to \infty$, $T \to 0$, reflecting the classical expectation that particles cannot pass through infinitely high barriers.

- When $V_0 \to \infty$, $T \to 0$, matching classical intuition.

Practical Implications and Applications

Quantum tunneling has profound implications in various physical and technological contexts:

- Nuclear Fusion: Tunneling allows nuclei to overcome Coulomb repulsion at energies where classical physics predicts they should not fuse.

- Semiconductor Devices: Tunnel diodes and quantum well transistors rely on tunneling effects for their operation.

- Scanning Tunneling Microscopes: Exploit electron tunneling between a sharp tip and a surface to image atomic-scale structures.

- Radioactive Decay: Certain decay processes proceed via tunneling of particles out of the nucleus.

Experimental Verification

The predictions of tunneling are not merely theoretical. Experiments have confirmed the finite transmission probabilities in various systems:

- Electron tunneling in metal-insulator-metal junctions exhibits behaviors matching quantum models.

- Observations of alpha decay in radioactive materials align with tunneling probability calculations.

- Semiconductor tunneling devices demonstrate the exponential dependence on barrier parameters predicted by theory.

Advanced Considerations

Multiple Barriers and Resonant Tunneling

- When particles encounter multiple barriers or potential wells, phenomena like resonant tunneling lead to sharp peaks in transmission probability at specific energies.

Time-Dependent Tunneling

- Time-dependent formulations address how quickly tunneling occurs and relate to the so-called "tunneling time"—a nuanced and debated topic.

Non-Rectangular Barriers

- More complex potential profiles require numerical methods but often reveal similar tunneling behavior, emphasizing the universality of the effect.

Conclusion

The phenomenon of quantum tunneling through a rectangular potential barrier exemplifies the departure from classical physics that quantum mechanics embodies. The key takeaway is that, for $V_0 > 0$ and $A > 0$

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