Consider The Differential Equation Y" + "' - 6y= 0. Part 1: Checking For Solutions (a) Which Of The Following

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When analyzing a second-order linear differential equation such as $Y'' + Y' - 6Y = 0$, one of the initial and crucial steps involves verifying potential solutions and understanding the structure of the solution space. This process helps determine whether certain functions qualify as solutions, whether the general solution can be expressed in closed form, and how to proceed with methods such as characteristic equations or variation of parameters.

In this article, we will delve into the process of checking for solutions to the differential equation $Y'' + Y' - 6Y = 0$. We will explore the approach to identifying particular solution types, understanding the implications of the differential operator, and evaluating specific candidate solutions. Part 1 focuses on determining which functions among a given set of options satisfy the differential equation, based on substitution and algebraic verification.

Understanding the Differential Equation: Basic Concepts

Form of the Differential Equation

The given differential equation:

$$
Y'' + Y' - 6Y = 0
$$

is a linear homogeneous differential equation with constant coefficients. Such equations are characterized by their coefficients being constants, simplifying the solution process. The general approach involves assuming solutions of a particular form (usually exponential functions) and deriving the characteristic equation.

Significance of Homogeneity and Linearity

Because the equation is homogeneous and linear, any linear combination of solutions is also a solution. This property simplifies the process of verifying candidate solutions—if a particular function satisfies the equation, it is a legitimate solution, and the entire solution space can be constructed from fundamental solutions.

Methodology for Checking Solutions

Substituting Candidate Solutions

To verify whether a specific function $y = y(x)$ is a solution, substitute it into the differential equation. This involves calculating the first and second derivatives of y and plugging these into the equation:

$$
Y'' + Y' - 6Y
$$

If the resulting expression simplifies to zero for all x, then y is a solution.

Common Types of Candidate Solutions

Candidates often include:

- Exponential functions: $y = e^{ax}$
- Polynomials: $y = x^n$
- Trigonometric functions: $y = \sin bx$ or $y = \cos bx$
- Linear functions: $y = mx + c$

In our case, initial options might include these types, or specific functions provided in multiple-choice options.

Checking Specific Candidate Solutions: Step-by-Step Process

Example 1: Exponential Solution $y = e^{rx}$

Suppose we consider $y = e^{rx}$. Then,

$$
Y' = r e^{rx}
$$
$$
Y'' = r^2 e^{rx}
$$

Substituting into the differential equation:

$$
r^2 e^{rx} + r e^{rx} - 6 e^{rx} = 0
$$

Dividing through by e^{rx} (which is never zero):

$$
r^2 + r - 6 = 0
$$

This quadratic equation in r:

$$
r^2 + r - 6 = 0
$$

can be solved using the quadratic formula:

$$
r = \frac{-1 \pm \sqrt{1 + 24}}{2} = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2}
$$

which yields:

$$
r = 2 \quad \text{or} \quad r = -3
$$

Thus, the exponential functions e^{2x} and e^{-3x} are solutions.

Implication: Any linear combination such as:

$$
y = C_1 e^{2x} + C_2 e^{-3x}
$$

will solve the differential equation.

Example 2: Polynomial Solution $y = x^n$

Try $y = x^n$, where n is a real number. Compute derivatives:

$$
Y' = n x^{n-1}
$$
$$
Y'' = n(n-1) x^{n-2}
$$

Substitute into the original equation:

$$
n(n-1) x^{n-2} + n x^{n-1} - 6 x^{n} = 0
$$

Divide through by x^{n-2} (assuming $x \neq 0$):

$$
n(n-1) + n x - 6 x^2 = 0
$$

This simplifies to a polynomial in x, which cannot be identically zero unless the coefficients vanish for all x. Since the last term involves x^2, this is only possible if the coefficients are zero, which cannot happen for all x. Therefore, polynomial solutions are generally not solutions to this differential equation unless trivial solutions are considered.

Example 3: Trigonometric Functions $y = \sin bx$ or $y = \cos bx$

Suppose $y = \sin bx$. Derivatives:

$$
Y' = b \cos bx
$$
$$
Y'' = -b^2 \sin bx
$$

Substitute into the differential equation:

\[

$$
(-b^2 \sin bx) + (b \cos bx) - 6 \sin bx = 0
$$

Rearranged:

$$
(-b^2 - 6) \sin bx + b \cos bx = 0
$$

For this to hold for all x, both coefficients must be zero:

$$
-b^2 - 6 = 0 \quad \Rightarrow \quad b^2 = -6
$$

$$
b = \pm i \sqrt{6}
$$

which is imaginary, therefore, real trigonometric solutions do not satisfy the differential equation unless complex solutions are considered.

Summary of Solution Checking Process

The process of checking whether a given function is a solution involves:

1. Calculating the derivatives of the candidate function.
2. Substituting derivatives into the differential equation.
3. Simplifying the resulting expression.
4. Verifying whether the expression reduces identically to zero for all x.

If the expression equals zero identically, the candidate function is a solution.

Implications for Multiple-Choice Options

When presented with multiple options, each candidate should be tested systematically:

- For exponential options, derive the characteristic roots.
- For polynomial options, analyze the derivatives and see if the expression simplifies to zero.
- For trigonometric options, check the necessary conditions for solutions, including complex roots if necessary.

The key is algebraic verification through substitution and simplification.

Conclusion

In the initial stage of analyzing the differential equation $Y'' + Y' - 6Y = 0$, the main task is to verify potential solutions through substitution. Recognizing that exponential functions form the basis of general solutions for linear equations with constant coefficients, the characteristic equation $r^2 + r - 6 = 0$ guides us to the fundamental solutions e^{2x} and e^{-3x}. Other candidate functions, such as polynomials or trigonometric functions, can be tested similarly to determine if they satisfy the equation, though typically, exponential solutions dominate the structure.

This process of verification not only clarifies which functions are solutions but also enhances understanding of the solution space's structure. Once the fundamental solutions are identified, the general solution can be expressed as a linear combination. This step is critical for solving initial value problems or boundary value problems associated with the differential equation.

In practice, this systematic approach to checking solutions is a vital skill for anyone studying differential equations, whether for theoretical analysis or applied modeling. The ability to quickly test candidate functions and understand the underlying structure of solutions forms the foundation for more advanced solution techniques, such as variation of parameters or Laplace transforms, which may be employed for more complex or nonhomogeneous equations.

Note: When working with multiple-choice options, always perform substitution checks to confirm solutions. For the specific differential equation $Y'' + Y' - 6Y = 0$, exponential functions corresponding to roots of the characteristic equation are the primary solutions, but other functions can sometimes be solutions depending on the context or boundary conditions.

Frequently Asked Questions

What is the characteristic equation associated with the differential equation Y'' - 6y = 0?

The characteristic equation is $r^2 - 6 = 0$.

How do you determine the roots of the characteristic equation for Y'' - 6y = 0?

Solve $r^2 - 6 = 0$, which gives $r = \pm\sqrt{6}$.

What form do the solutions take when the roots of the characteristic equation are real and distinct?

The solutions are of the form $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$, where r_1 and r_2 are the roots.

Are exponential functions valid solutions to the differential equation Y'' - 6y = 0?

Yes, exponential functions of the form e^{rt} are solutions when r satisfies the characteristic equation.

What is the significance of checking potential solutions like y = e^{kt} when solving the differential equation?

It helps verify whether exponential functions are solutions by substituting into the differential equation to find compatible values of k.

Can polynomial or sinusoidal functions be solutions to Y'' - 6y = 0?

No, because substituting such functions does not satisfy the differential equation unless they are linear combinations of exponential functions derived from the roots.

What is the general solution to the differential equation Y'' - 6y = 0?

The general solution is $y(t) = C_1 e^{\sqrt{6} t} + C_2 e^{-\sqrt{6} t}$.

How do you check if a proposed solution y = e^{kt} is valid for the differential equation?

Substitute $y = e^{kt}$ into the differential equation and verify whether the resulting expression holds true for specific values of k.

What are the key steps in Part 1 of solving the differential equation: checking for solutions?

Identify the form of potential solutions, derive the characteristic equation, find its roots, and verify the solutions satisfy the differential equation.

Additional Resources

Consider The Differential Equation Y'' + Y''' - 6y = 0: Part 1 – Checking For Solutions (a) Which Of The Following

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Introduction: The Significance of Differential Equations in Mathematical Modeling

Differential equations serve as fundamental tools in mathematics, physics, engineering, and numerous applied sciences, enabling us to describe systems where quantities change continuously over time or space. Among these, linear homogeneous differential equations with constant coefficients are particularly crucial due to their tractability and extensive applicability. In this context, analyzing the solutions of such equations not only deepens our understanding of the underlying phenomena but also lays the groundwork for more complex, nonlinear models.

This article delves into a specific third-order linear differential equation:
\[
Y''' + Y'' - 6Y = 0
\]
with a focus on determining which among a set of proposed solutions satisfy the equation, thereby serving as valid solutions. The process involves examining the characteristic equation, understanding the nature of roots, and applying algebraic techniques to check the validity of candidate solutions.

Understanding the Differential Equation: Structure and Key Features

The Equation at a Glance

The equation under consideration is a third-order linear homogeneous differential equation with constant coefficients:
\[
Y''' + Y'' - 6Y = 0
\]
where \(Y'''\) denotes the third derivative of \(Y\) with respect to the independent variable (often time \(t\)), and similarly for \(Y''\) and \(Y\).

The general form of such an equation can be expressed as:
\[

$$
a_3 Y''' + a_2 Y'' + a_1 Y' + a_0 Y = 0
$$
where a_3, a_2, a_1, a_0 are constants. In our case:
$$
a_3 = 1, \quad a_2 = 1, \quad a_1 = 0, \quad a_0 = -6
$$

Why is this important? Because linear equations with constant coefficients allow us to employ the characteristic equation method to find solutions systematically.

Characteristic Equation: The Key to Solutions

The solution process hinges on forming and solving the characteristic equation associated with the differential equation. For the general third-order linear homogeneous differential equation, the characteristic equation is:
$$
a_3 r^3 + a_2 r^2 + a_1 r + a_0 = 0
$$

Applying this to our specific equation yields:
$$
r^3 + r^2 - 6 = 0
$$

The roots of this polynomial determine the form of the general solution, whether they are real, repeated, or complex conjugates.

Part 1: Checking For Solutions – The Approach

Step 1: Derive the Characteristic Equation and Find Its Roots

The first step involves solving the cubic polynomial:
$$
r^3 + r^2 - 6 = 0
$$

This is crucial because the nature of these roots—real and distinct, real and repeated, or complex conjugate—directly influences the form of the solutions to the differential equation.

Step 2: Identify Candidate Solutions

Once the roots are known, typical solutions to the differential equation are formulated based on root types:

- Three distinct real roots r_1, r_2, r_3:

$$
y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + C_3 e^{r_3 t}
$$

- Repeated roots:
For multiplicity m, include polynomial factors multiplied by exponentials.

- Complex roots $a \pm bi$:

$$
y(t) = e^{a t} (A \cos b t + B \sin b t)
$$

The candidate solutions provided in the problem are typically functions of this form, and part of our task is to verify which of these satisfy the original differential equation.

Step 3: Substitution and Verification

To verify whether a candidate function $Y(t)$ is a solution, substitute it into the differential equation:

$$
Y''' + Y'' - 6Y
$$

and check if the expression simplifies to zero. If it does, the candidate is a valid solution; if not, it is invalid.

Part 2: Solving the Characteristic Equation

Solving the Cubic Polynomial: $r^3 + r^2 - 6 = 0$

The cubic polynomial is not factorable easily at first glance, but common techniques include:

- Rational Root Theorem
- Polynomial division
- Synthetic division
- Factoring by inspection

Rational Root Theorem:

Possible rational roots are factors of the constant term -6, over factors of the leading coefficient 1:

$$
\pm 1, \pm 2, \pm 3, \pm 6
$$

We test these values:

- For $r=1$:
$$
1^3 + 1^2 - 6 = 1 + 1 - 6 = -4 \neq 0
$$
- For $r=-1$:
$$
(-1)^3 + (-1)^2 - 6 = -1 + 1 - 6 = -6 \neq 0
$$
- For $r=2$:
$$
8 + 4 - 6 = 6 \neq 0
$$
- For $r=-2$:
$$
-8 + 4 - 6 = -10 \neq 0
$$
- For $r=3$:
$$
27 + 9 - 6 = 30 \neq 0
$$
- For $r=-3$:
$$
-27 + 9 - 6 = -24 \neq 0
$$
- For $r=6$:
$$
216 + 36 - 6= 246 \neq 0
$$
- For $r=-6$:
$$
-216 + 36 - 6= -186 \neq 0
$$

None of these are roots; therefore, the roots are irrational or complex.

Next step: Use depressed cubic techniques or numerical methods to approximate roots or factor further.

Alternatively, employ a substitution or Cardano's method to find roots explicitly.

Part 3: Analytical Solution of the Cubic Equation

Applying Cardano's Method

The depressed cubic form is:

$$
r^3 + p r + q = 0
$$

where, in our case:

$$
r^3 + r^2 - 6 = 0
$$

To convert to depressed form, set $r = s - \frac{b}{3a}$, but because the original cubic is in the form $r^3 + r^2 - 6 = 0$, we can perform substitution to eliminate the quadratic term.

Let's rewrite:

$$
r^3 + r^2 - 6 = 0
$$

Divide the entire equation by 1 (since leading coefficient is 1):

$$
r^3 + r^2 - 6 = 0
$$

Set $r = t - \frac{1}{3}$ to remove the quadratic term:

$$
(t - \frac{1}{3})^3 + (t - \frac{1}{3})^2 - 6 = 0
$$

Expanding:

$$
t^3 - t^2 + \frac{1}{3} t - \frac{1}{27} + t^2 - \frac{2}{3} t + \frac{1}{9} - 6 = 0
$$

Simplify:

$$
t^3 + \left(- t^2 + t^2\right) + \left(\frac{1}{3} t - \frac{2}{3} t\right) + \left(- \frac{1}{27} + \frac{1}{9} - 6\right) = 0

\]

which reduces to:

\[
t^3 - \frac{1}{3} t + \left(- \frac{1}{27} + \frac{3}{27} - \frac{162}{27}\right) = 0
\]

Calculate the constant:

\[
- \frac{1}{27} + \frac{3}{27} - \frac{162}{27} = \frac{-1 + 3 - 162}{27} = \frac{-160}{27}
\]

The depressed cubic becomes:

\[
t^3 - \frac{1}{3} t - \frac{160}{27} = 0
\]

Now, applying Cardano's formula involves calculating:

\[
\Delta = \left(\frac{q}{2} \right)^2 + \left(\frac{p}{3} \right)^3
\]

where:

\[
p = - \frac{1}{

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