Find The Dimensions Of A Rectangle With A Perimeter 60m Whose Area Is As Large As Possible. Area Of Rectangle

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Understanding how to optimize the dimensions of a rectangle for maximum area given a fixed perimeter is a fundamental problem in geometry and calculus. This problem not only illustrates important mathematical principles but also has practical applications in fields such as architecture, engineering, and landscaping. In this article, we will explore how to find the dimensions of a rectangle with a perimeter of 60 meters that yields the largest possible area. We will analyze the problem step-by-step, employ mathematical formulas, and demonstrate the solution process through calculus techniques to maximize the rectangle’s area.

Understanding the Problem: Rectangles and Perimeter Constraints

Before delving into calculations, it is essential to understand the key components of the problem:

- Perimeter of the rectangle (P): The total length around the rectangle. In this case, P = 60 meters.
- Area of the rectangle (A): The space enclosed within the rectangle, which we aim to maximize.
- Dimensions of the rectangle: Length (L) and width (W).

The goal is to determine the specific values of L and W that produce the maximum area, given the perimeter constraint.

Mathematical Formulation of the Problem

The problem can be mathematically expressed as follows:

- Perimeter formula for a rectangle:

$$
P = 2L + 2W
$$

Given that $P = 60$ meters:

$$
2L + 2W = 60
$$

- Area formula:

$$
A = L \times W
$$

Our task is to find values of L and W that maximize A, subject to the perimeter constraint.

Expressing One Variable in Terms of the Other

To simplify the problem, express one dimension in terms of the other using the perimeter constraint:

$$
2L + 2W = 60
$$

Dividing both sides by 2:

$$
L + W = 30
$$

Express W in terms of L:

$$
W = 30 - L
$$

Now, the area A becomes a function of a single variable:

$$
A(L) = L \times W = L \times (30 - L)
$$

$$
A(L) = 30L - L^2
$$

\]

This quadratic function describes the area in terms of \(L \).

Maximizing the Area: Calculus Approach

To find the maximum area, differentiate \(A(L) \) with respect to \(L \) and set the derivative equal to zero:

\[
A'(L) = \frac{d}{dL}(30L - L^2) = 30 - 2L
\]

Set the derivative equal to zero to find critical points:

\[
30 - 2L = 0
\]

\[
2L = 30
\]

\[
L = 15
\]

Since \(W = 30 - L \):

\[
W = 30 - 15 = 15
\]

Thus, the rectangle with the maximum area, given the perimeter constraint, has dimensions:

\[
L = 15\, \text{meters}, \quad W = 15\, \text{meters}
\]

This indicates that the rectangle is actually a square.

Verifying the Maximum Area

To confirm that this critical point corresponds to a maximum, examine the second derivative:

$$
A''(L) = \frac{d^2}{dL^2}(30L - L^2) = -2
$$

Since $A''(L) = -2 < 0$, the function is concave down at $L = 15$, confirming a maximum.

Maximum Area Calculation:

$$
A_{max} = L \times W = 15 \times 15 = 225\, \text{m}^2
$$

Therefore, the largest possible area of the rectangle with a perimeter of 60 meters is 225 square meters, achieved when the rectangle is a square with sides of 15 meters.

Additional Insights and Practical Applications

Understanding that a square provides the maximum area for a given perimeter is a fundamental geometric principle. This insight has various practical applications:

- Design and Architecture: To maximize usable space within a fixed boundary, designing structures as squares optimizes area.
- Agriculture and Land Planning: When fencing a land plot with a fixed length of fencing (perimeter), making the plot square maximizes the enclosed area.
- Material Efficiency: In manufacturing, choosing dimensions that maximize area can reduce material wastage and improve efficiency.

Summary of Key Steps to Find the Dimensions

Here's a quick overview of the methodology used:

1. Write the perimeter constraint: $2L + 2W = 60$.
2. Express one variable in terms of the other: $W = 30 - L$.

3. Formulate the area function: $A(L) = L(30 - L)$.
4. Differentiate and find critical points: $A'(L) = 30 - 2L = 0$.
5. Solve for L: $L = 15$.
6. Calculate W: $W = 15$.
7. Verify the maximum: Use the second derivative test; since it's negative, the critical point is a maximum.
8. Compute maximum area: $15 \times 15 = 225\, \text{m}^2$.

Conclusion

In summary, the rectangle with a perimeter of 60 meters that has the largest possible area is a square with side lengths of 15 meters, yielding an area of 225 square meters. This result aligns with geometric principles stating that among rectangles with a fixed perimeter, the square has the maximum area. This problem demonstrates the power of calculus in optimizing geometric figures and provides valuable insights applicable in various real-world contexts where maximizing space efficiency is essential.

Key Takeaways:

- The maximum area of a rectangle with a fixed perimeter occurs when the rectangle is a square.
- Mathematical techniques such as substitution and differentiation are effective tools for optimization problems.
- Understanding these principles helps in designing efficient, space-maximizing structures and layouts.

Meta Description: Discover how to find the dimensions of a rectangle with a perimeter of 60 meters that maximizes its area. Learn step-by-step calculus methods to optimize rectangle dimensions for maximum space.

Frequently Asked Questions

What is the goal when finding the dimensions of a rectangle with a perimeter of 60 meters?

The goal is to determine the length and width that maximize the area of the rectangle while maintaining a perimeter of 60 meters.

How do you express the area of a rectangle in terms of its sides?

The area of a rectangle is expressed as length multiplied by width, i.e., Area = length × width.

What is the formula for the perimeter of a rectangle?

The perimeter of a rectangle is given by the formula P = 2(length + width).

How can we set up an equation for the dimensions given a perimeter of 60 meters?

Since P = 2(L + W) and P = 60, we have 2(L + W) = 60, which simplifies to L + W = 30.

How do we maximize the area of the rectangle under the given perimeter constraint?

Express one variable in terms of the other using L + W = 30, then write the area as a function of one variable and find its maximum using calculus or algebraic methods.

What are the optimal dimensions of the rectangle for maximum area?

The maximum area occurs when the rectangle is a square, with both sides equal to 15 meters, i.e., L = W = 15m.

What is the maximum area of the rectangle with a perimeter of 60 meters?

The maximum area is 15m × 15m = 225 square meters.

Why does the rectangle with maximum area turn out to be a square in this problem?

Because for a given perimeter, a square encloses the largest possible area among rectangles, as proven by the isoperimetric principle.

How can this problem be generalized to find the maximum area of rectangles with different

perimeters?

By setting the perimeter constraint and expressing the area as a function of one variable, then using calculus or algebra to find the maximum, similar to the steps in this specific problem.

Additional Resources

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Introduction

In the realm of geometry and optimization, the problem of determining the dimensions of a rectangle with a fixed perimeter that maximizes its area is a classical yet fascinating challenge. Whether in engineering, architecture, or everyday problem-solving, understanding how to optimize the area within given constraints is essential. This article delves deep into this problem, providing a comprehensive exploration of the mathematical principles involved, the derivation of solutions, and practical implications.

Understanding the Problem

The Core Question

Given a rectangle with a perimeter of 60 meters, what are its length and width such that the area is maximized?

This question involves two fundamental geometric concepts:

- Perimeter: The total length of the boundary of the rectangle.
- Area: The space enclosed within the rectangle.

The goal is to find the dimensions (length and width) that optimize the area while respecting the perimeter constraint.

Significance of the Problem

Maximizing the area for a given perimeter is not merely an academic exercise. It reflects real-world scenarios such as:

- Designing enclosures with limited fencing material.
- Planning agricultural plots for maximum cultivation space.
- Developing efficient layouts in manufacturing or construction.

Understanding how to approach such problems enhances decision-making skills in various fields.

Mathematical Foundations

Defining Variables

Let:

- L = Length of the rectangle (meters)
- W = Width of the rectangle (meters)

Given the perimeter:

$$
P = 2L + 2W = 60 \text{ meters}
$$

We seek to maximize the area:

$$
A = L \times W
$$

Expressing Area in Terms of a Single Variable

To optimize the area, it's convenient to express A as a function of a single variable.

From the perimeter constraint:

$$
2L + 2W = 60
$$
$$
L + W = 30
$$
$$
W = 30 - L
$$

Substituting into the area formula:

$$
A(L) = L \times (30 - L) = 30L - L^2
$$

This quadratic function describes how the area varies with the length L, with the width derived accordingly.

Analytical Solution

Maximizing the Area Function

The function:

$$
A(L) = -L^2 + 30L
$$

is a quadratic with a negative leading coefficient, indicating a parabola opening downward, and thus has a maximum point at its vertex.

Finding the Vertex

The vertex of a quadratic $ax^2 + bx + c$ occurs at:

$$
L = -\frac{b}{2a}
$$

In our case:

- $a = -1$
- $b = 30$

Therefore:

$$
L_{max} = -\frac{30}{2 \times -1} = -\frac{30}{-2} = 15
$$

Correspondingly, the width:

$$
W = 30 - L = 30 - 15 = 15
$$

Interpretation

The maximum area is achieved when length and width are both 15 meters, which suggests a square.

Calculating the Maximum Area

$$
A_{max} = 15 \times 15 = 225 \text{ square meters}
$$

Geometric and Practical Significance

The Square as the Optimal Shape

The derivation reveals a fundamental geometric principle: for a given perimeter, the rectangle with the largest area is a square. This aligns with well-known mathematical facts and has practical validation in various fields.

Why Is the Square Optimal?

- Symmetry: Squares maximize enclosed area for a given perimeter because all sides are equal, distributing the boundary evenly.
- Mathematical Confirmation: Calculus-based optimization confirms that the maximum occurs at equal sides.

Real-world Constraints and Variations

While the ideal solution suggests a perfect square, real-world applications might involve constraints such as:

- Non-square land plots due to terrain or existing structures.
- Variations in fencing costs for different sides.
- Regulatory or zoning restrictions.

In such cases, the solution provides a benchmark, and deviations can be analyzed accordingly.

Alternative Approaches and Validations

Using Calculus

The derivative of $A(L)$:

$$
A'(L) = 30 - 2L
$$

Setting to zero for maximum:

$$
30 - 2L = 0 \Rightarrow L = 15
$$

This confirms the earlier result.

Graphical Interpretation

Plotting $A(L)$:

- The parabola peaks at $L=15$,
- The maximum area at the vertex is 225 m^2.

Confirming with Boundary Conditions

- $L = 0$, $W=30$ – Area = 0 (degenerate rectangle).
- $L=30$, $W=0$ – Area = 0.
- The maximum at $L=15$, $W=15$.

Practical Applications and Implications

Fencing and Enclosure Design

Suppose a farmer has 60 meters of fencing and wants to enclose a field with the maximum possible area. The solution indicates that constructing a square plot measuring 15 meters on each side will optimize land use efficiency.

Urban Planning and Architecture

Designers often aim for square or near-square layouts to maximize space utilization within perimeter constraints, whether for parks, playgrounds, or building footprints.

Manufacturing and Material Optimization

In manufacturing processes involving sheets or panels, understanding how to maximize surface area within boundary limits informs resource allocation.

Limitations and Considerations

While the mathematical model provides an ideal solution, several factors might influence real-world implementation:

- Material limitations: Fencing materials may have costs or constraints.
- Terrain irregularities: Not all land is perfectly flat or rectangular.
- Accessibility and aesthetics: Practical considerations might lead to deviations from the optimal shape.
- Additional constraints: Zoning laws, environmental restrictions, or existing infrastructure.

Understanding the theoretical maximum serves as a benchmark, guiding practical decision-making.

Extending the Concept

Other Perimeter Constraints

The approach can be generalized to different perimeter lengths or shapes:

- For a fixed perimeter P, the maximum area is always achieved by a square with side $P/4$.
- For irregular shapes, optimization involves more complex calculus or computational methods.

Three-Dimensional Analogues

The concept extends into three dimensions, such as maximizing volume within a fixed surface area or boundary length, leading to shapes like spheres or cubes as optimal solutions.

Conclusion

The problem of determining the dimensions of a rectangle with a fixed perimeter that maximizes its area encapsulates fundamental principles of geometry, calculus, and optimization. Through algebraic and calculus-based methods, it is demonstrated that a square configuration yields the maximum area, which, in this case, is 225 square meters for a 60-meter perimeter.

This exploration underscores the elegance of mathematical principles in solving real-world problems and highlights the importance of symmetry and geometric optimization in practical design and planning. Whether for fencing a garden, designing urban spaces, or manufacturing components, understanding these concepts offers valuable insights into efficient resource utilization and spatial planning.

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