If X Has An Exponential Distribution With Parameter , Derive A General Expression For The (100p)th Percentile

Introduction

If X Has An Exponential Distribution With Parameter λ, Derive A General Expression For The (100p)th Percentile

The exponential distribution is a widely used continuous probability distribution, particularly in modeling the times between events in a Poisson process. Its simplicity and memoryless property make it a popular choice in various fields such as reliability engineering, queuing theory, and survival analysis. When working with the exponential distribution, one often needs to determine specific quantiles or percentiles, which provide critical insights into the distribution's behavior. In this article, we focus on deriving a general expression for the (100p)th percentile of an exponential distribution with parameter λ, where p is a probability value between 0 and 1. We will explore the derivation step-by-step, discuss its implications, and provide practical examples to illustrate the concept.

Understanding the Exponential Distribution

Definition and Probability Density Function (PDF)

The exponential distribution is characterized by its rate parameter λ > 0. Its probability density function (PDF) is given by:

- **PDF:** $f_X(x) = \lambda e^{-\lambda x}$, for $x \geq 0$

The distribution is supported on the interval $[0, \infty)$, meaning that the random variable X takes non-negative values.

Cumulative Distribution Function (CDF)

The cumulative distribution function (CDF), which gives the probability that X takes a value less than or equal to x, is derived by integrating the PDF:

- **CDF:** $F_X(x) = P(X \leq x) = 1 - e^{-\lambda x}$, for $x \geq 0$

This function is monotonically increasing from 0 to 1 as x goes from 0 to infinity.

Deriving the (100p)th Percentile

Definition of a Percentile

The (100p)th percentile, denoted as x_p, is the value of X such that the probability of X being less than or equal to x_p is p. Mathematically, this is expressed as:

- $P(X \leq x_p) = p$

Equivalently, since $F_X(x) = 1 - e^{-\lambda x}$, we have:

- $F_X(x_p) = p$

Step-by-Step Derivation

1. Start with the definition of the CDF at the percentile point:
$$
F_X(x_p) = p
$$

2. Substitute the exponential CDF:
$$
1 - e^{-\lambda x_p} = p
$$

3. Isolate the exponential term:
$$
e^{-\lambda x_p} = 1 - p
$$

4. Take the natural logarithm on both sides:
$$
-\lambda x_p = \ln(1 - p)
$$

5. Finally, solve for x_p:
$$
x_p = -\frac{1}{\lambda} \ln(1 - p)
$$

General Expression for the (100p)th Percentile

Based on the derivation above, the general formula for the (100p)th percentile of an exponential distribution with rate parameter λ is:

Expression:

$$
\boxed{
x_p = -\frac{1}{\lambda} \ln(1 - p), \quad 0 < p < 1
}
$$

This formula provides a direct way to compute any percentile given a probability p and the parameter λ.

Special Cases and Applications

Median of the Exponential Distribution

When $p = 0.5$, the formula simplifies to the median:

$$
x_{0.5} = -\frac{1}{\lambda} \ln(1 - 0.5) = -\frac{1}{\lambda} \ln(0.5) = \frac{\ln 2}{\lambda}
$$

This indicates that the median is a fixed multiple of $1/\lambda$, which depends solely on the rate parameter.

Quantile Function (Inverse CDF)

The derived expression is also known as the inverse CDF or quantile function of the exponential distribution, allowing generation of specific quantiles or simulation of exponential random variables given a uniform random variable U ~ Uniform(0,1):

- Set $U = p$, then:
  $$
  X = -\frac{1}{\lambda} \ln(1 - U)
  $$

This is a common technique in Monte Carlo simulations for generating exponential variates.

Practical Examples

Example 1: Computing the 90th Percentile

Suppose $\lambda = 2$. To find the 90th percentile ($p=0.9$), apply the formula:

$$
x_{0.9} = -\frac{1}{2} \ln(1 - 0.9) = -\frac{1}{2} \ln(0.1)
$$

Calculating numerically:

$$
x_{0.9} \approx -0.5 \times (-2.302585) = 1.1513
$$

Thus, 90% of the values are less than approximately 1.15 units.

Example 2: Generating a Random Variable with a Given Percentile

Assuming a uniform random variable $U \sim \text{Uniform}(0, 1)$, we can generate an exponential random variable X with rate λ such that X exceeds the pth percentile with probability $1 - p$:

- Set $p = 0.95$, then compute:
  $$
  X = -\frac{1}{\lambda} \ln(1 - U)
  $$
- Interpretation: When $U=0.95$, the generated X corresponds exactly to the 95th percentile.

Implications and Uses of the Percentile Formula

Risk Analysis and Reliability Engineering

The ability to compute specific percentiles allows engineers and statisticians to estimate the worst-case or extreme scenarios, such as the maximum expected lifetime or failure time within a certain confidence level.

Data Simulation and Modeling

Simulation techniques often require generating random samples from the exponential distribution. Using the inverse CDF method, which relies on the derived percentile formula, facilitates efficient and accurate simulation of exponential variates.

Comparison with Other Distributions

While many distributions have complex quantile functions, the exponential distribution's inverse CDF has a simple closed-form expression, making it especially convenient in both theoretical and applied contexts.

Summary

In conclusion, for an exponential distribution with rate parameter λ, the (100p)th percentile x_p is given by:

$$
x_p = -\frac{1}{\lambda} \ln(1 - p)
$$

This compact formula provides a straightforward way to find any percentile, interpret probabilistic thresholds, and generate random variates via inverse transform sampling. Its simplicity underscores the exponential distribution's usefulness in modeling waiting times and other phenomena characterized by constant hazard rates.

References

- Ross, S. M. (2014). *Introduction to Probability Models*. Academic Press.
- Casella, G., & Berger, R. L. (2002). *Statistical Inference*. Duxbury Advanced Series.
- Devroye, L. (1986). *Non-Uniform Random Variate Generation*. Springer.

Frequently Asked Questions

What is the general formula for the p-th percentile of an exponential distribution with parameter λ?

The p-th percentile, denoted as Q_p, for an exponential distribution with rate λ is given by $Q_p = -(1/\lambda) \ln(1 - p)$.

How do you derive the (100p)th percentile for an exponential distribution with parameter λ?

Starting from the cumulative distribution function (CDF) $F(x) = 1 - e^{-\lambda x}$, we set $F(Q_p) = p$ and solve for Q_p: $Q_p = -(1/\lambda) \ln(1 - p)$.

Why is the exponential distribution's p-th percentile expressed as - (1/λ) ln(1 - p)?

Because the exponential distribution's CDF is $F(x) = 1 - e^{-\lambda x}$, solving $F(x) = p$ for x yields $x = -(1/\lambda) \ln(1 - p)$, which defines the p-th percentile.

What is the significance of the (100p)th percentile in the context of exponential distributions?

The (100p)th percentile indicates the value below which a proportion p of the data falls, useful for understanding the distribution's behavior and for threshold-based decision making.

Can the formula for the (100p)th percentile be applied to any exponential distribution with a different rate parameter λ?

Yes, the formula $Q_p = -(1/\lambda) \ln(1 - p)$ applies universally to exponential distributions with any positive rate λ, making it a versatile tool for percentile calculations.

Additional Resources

If X Has An Exponential Distribution With Parameter λ, Derive a General Expression For The (100p)th Percentile

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Introduction

In the realm of probability distributions, the exponential distribution holds a significant place due to its simplicity and applicability across various domains such as reliability engineering, queuing theory, survival analysis, and more. Its fundamental property of modeling the time between independent events occurring at a constant average rate makes it a pivotal distribution in stochastic processes.

A key aspect of understanding any distribution involves quantile analysis—specifically, the calculation of percentiles. Percentiles provide insights into the distribution's spread, central tendency, and tail behavior, which are essential for decision-making, risk assessment, and statistical inference.

This article aims to rigorously derive a general expression for the (100p)th percentile of an exponential distribution with parameter λ. We will explore the distribution's properties, define the percentile concept, and systematically derive the formula, ensuring clarity and thoroughness for readers with a foundational knowledge of probability theory.

The Exponential Distribution: An Overview

The exponential distribution with parameter $\lambda > 0$ is characterized by its probability density function (PDF) and cumulative distribution function (CDF):

- Probability Density Function (PDF):

$$
f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0
$$

- Cumulative Distribution Function (CDF):

$$
F_X(x) = P(X \leq x) = 1 - e^{-\lambda x}, \quad x \geq 0
$$

The exponential distribution is memoryless, meaning that:

$$
P(X > s + t \mid X > s) = P(X > t)
$$

for all $s, t \geq 0$.

Understanding Percentiles in Probability Distributions

A percentile of a distribution provides the value below which a certain percentage p (where $0 < p < 1$) of the data falls. Formally, the (100p)th percentile of a random variable X, denoted as x_p, is the value satisfying:

$$
P(X \leq x_p) = p
$$

or equivalently,

$$
F_X(x_p) = p
$$

Deriving the general expression for x_p involves solving the equation:

$$
F_X(x_p) = p
$$

for x_p.

Derivation of the (100p)th Percentile for the Exponential Distribution

Given the CDF of the exponential distribution:

$$
F_X(x) = 1 - e^{-\lambda x}
$$

the goal is to find x_p such that:

$$
F_X(x_p) = p
$$

which leads to:

$$
1 - e^{-\lambda x_p} = p
$$

Rearranged, this becomes:

$$
e^{-\lambda x_p} = 1 - p
$$

To solve for x_p, take the natural logarithm on both sides:

$$
-\lambda x_p = \ln(1 - p)
$$

which yields:

$$
x_p = - \frac{1}{\lambda} \ln(1 - p)
$$

This formula provides the general expression for the (100p)th percentile of an exponential distribution with parameter λ.

Properties and Implications of the Percentile Formula

The derived formula:

$$
\boxed{
x_p = - \frac{1}{\lambda} \ln(1 - p)
}
$$

has several important properties:

1. Monotonicity: As p increases from 0 to 1, x_p increases from 0 to infinity, consistent with the distribution's support over $[0, \infty)$.

2. Percentile Calculation: For common percentiles:
- Median ($p = 0.5$):

$$
x_{0.5} = - \frac{1}{\lambda} \ln(1 - 0.5) = - \frac{1}{\lambda} \ln(0.5) = \frac{\ln 2}{\lambda}
$$

- 90th percentile ($p = 0.9$):

$$
x_{0.9} = - \frac{1}{\lambda} \ln(0.1)
$$

3. Asymptotic Behavior: As $p \to 1$, $x_p \to \infty$, reflecting the tail behavior of the exponential distribution.

4. Inverse Relationship: The formula underscores the inverse relationship between the percentile and the rate parameter λ; higher λ (faster decay) leads to smaller percentiles.

Special Cases and Practical Applications

1. Median of the Distribution

The median $x_{0.5}$:

$$
x_{0.5} = \frac{\ln 2}{\lambda}
$$

is often used as a measure of central tendency in exponential models when the mean is skewed.

2. Confidence Intervals for Percentiles

Using the formula, practitioners can compute various percentiles directly, which is critical in fields like reliability engineering (e.g., estimating the time by which a certain percentage of components fail).

3. Quantile-Based Risk Assessment

In survival analysis, the (100p)th percentile represents a time threshold below which a certain proportion of subjects are expected to experience an event, guiding decision-making and policy formulation.

Extending the Result: Generalized Percentile Expressions

While the derivation centers on the exponential distribution, the methodology exemplifies a broader principle:

- For any distribution with cumulative distribution function $F_X(x)$, the p-th percentile is obtained by solving:

$$
x_p = F_X^{-1}(p)
$$

where F_X^{-1} is the inverse CDF (quantile function). For the exponential distribution, this inverse is explicitly:

$$
F_X^{-1}(p) = - \frac{1}{\lambda} \ln(1 - p)
$$

highlighting the distribution's simplicity and analytical tractability.

Summary and Conclusions

In this review, we have:

- Clarified the concept of percentiles within probability distributions.
- Presented the probability density and cumulative functions of the exponential distribution.
- Systematically derived the general formula for the $(100p)$th percentile:

$$
\boxed{
x_p = - \frac{1}{\lambda} \ln(1 - p)
}
$$

- Discussed properties, special cases, and practical implications of this formula.

This derivation not only enhances understanding of the exponential distribution's quantile structure but also exemplifies the broader utility of inverse functions in probability theory. The explicit expression facilitates rapid computation of key percentiles, aiding practitioners and researchers in diverse fields where exponential models are applicable.

References

- Ross, S. M. (2014). Introduction to Probability Models. Academic Press.
- Johnson, N. L., Kotz, S., & Balakrishnan, N. (1994). Continuous Univariate Distributions. Wiley.
- Ross, S. M. (2010). A First Course in Probability. Pearson Education.

Note: The derivation assumes \(0 < p < 1 \) and \(\lambda > 0 \), consistent with the properties of the exponential distribution.

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