In Exercises 20-25, Find The Standard Matrix Of The Linear Transformation From R2 To R2. 20. Counterclockwise

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Understanding the standard matrix of a linear transformation is fundamental in linear algebra, especially when dealing with transformations from R^2 to R^2. Specifically, rotations, reflections, and other geometric transformations can be represented efficiently through matrices. Exercise 20 focuses on a classic transformation: counterclockwise rotation. This article aims to explore this transformation in depth, providing a comprehensive guide to determine the standard matrix for a counterclockwise rotation in R^2, along with contextual explanations, step-by-step procedures, and practical applications.

Introduction to Linear Transformations in R^2

What Is a Linear Transformation?

A linear transformation is a function $T: R^2 \to R^2$ that preserves vector addition and scalar multiplication. In mathematical terms, for vectors $\mathbf{u}, \mathbf{v} \in R^2$ and scalar c,

$$
T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \quad \text{and} \quad T(c\mathbf{u}) = cT(\mathbf{u})
$$

Linear transformations are essential because they describe a wide array of geometric operations such as rotations, reflections, scalings, and shears.

Standard Matrix of a Linear Transformation

Any linear transformation $T: R^2 \to R^2$ can be represented by a 2×2 matrix A, called the standard matrix of T. If $\mathbf{v}$ is a vector in R^2, then

$$
T(\mathbf{v}) = A \mathbf{v}
$$

The columns of A are the images of the basis vectors $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$.

Rotation in R^2: The Counterclockwise Rotation

Geometric Intuition

Rotation is a fundamental geometric transformation that turns every point in the plane around the origin by a specified angle θ. When a point $\mathbf{v} = (x, y)$ is rotated counterclockwise by an angle θ, its position changes according to the angle's measure, but the distance from the origin remains constant.

Mathematical Representation of Rotation

The rotation transformation, denoted as R_θ, rotates vectors by an angle θ counterclockwise. Its effect on a vector $\mathbf{v} = (x, y)$ can be described using trigonometric functions:

$$
R_\theta(\mathbf{v}) =
\begin{bmatrix}
x' \\ y'
\end{bmatrix}
=
\begin{bmatrix}
x \cos \theta - y \sin \theta \\[6pt]
x \sin \theta + y \cos \theta
\end{bmatrix}
$$

This formula is derived from the standard rotation of axes in Euclidean geometry.

Finding the Standard Matrix for Counterclockwise Rotation

Step-by-Step Process

To find the standard matrix corresponding to a counterclockwise rotation by θ:

1. Identify the effect on basis vectors:

- Apply the rotation R_θ to $\mathbf{e}_1 = (1, 0)$.
- Apply the rotation R_θ to $\mathbf{e}_2 = (0, 1)$.

2. Compute the images of the basis vectors:

- $R_\theta(\mathbf{e}_1) = (\cos \theta, \sin \theta)$

- $R_\theta(\mathbf{e}_2) = (-\sin \theta, \cos \theta)$

3. Construct the matrix:

- The columns of the matrix are the images of $\mathbf{e}_1$ and $\mathbf{e}_2$.

- Therefore, the standard matrix A for rotation by θ is:

$$
A =
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
$$

This matrix directly encodes the rotation transformation in a form suitable for matrix multiplication.

Example: Rotation by 45 Degrees

Suppose we want to find the standard matrix for a rotation of 45° (or $\pi/4$ radians):

- $\cos 45^\circ = \frac{\sqrt{2}}{2}$
- $\sin 45^\circ = \frac{\sqrt{2}}{2}$

Plugging into the matrix:

$$
A =
\begin{bmatrix}
$$

$$
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}
\end{bmatrix}
$$

This matrix rotates any vector in R^2 by 45 degrees counterclockwise.

Applications of Rotation Matrices

Geometric Transformations

Rotation matrices are used in computer graphics, robotics, physics, and engineering to model and simulate real-world rotations.

Computer Graphics and Animation

- Rotating images or objects in 2D space.
- Implementing rotational motion in animations.

Robotics

- Controlling the orientation of robotic arms.
- Calculating movement paths involving rotation.

Physics

- Describing the rotation of particles and rigid bodies.
- Analyzing angular momentum and rotational dynamics.

Properties of Rotation Matrices

Orthogonality

Rotation matrices are orthogonal matrices, meaning:

$$
A^T A = AA^T = I
$$

where A^T is the transpose of A, and I is the identity matrix.

Determinant

The determinant of a rotation matrix is always 1:

$$
\det(A) = 1
$$

which indicates that rotation preserves area and orientation in the plane.

Inverse

The inverse of a rotation matrix corresponds to rotation by the negative of the original angle:

$$
A^{-1} = R_{-\theta}
$$

which is the transpose of the rotation matrix:

$$
A^{-1} = A^T
$$

Summary

To summarize, the standard matrix for a counterclockwise rotation by an angle θ in R^2 is:

$$
\boxed{
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
}
$$

This matrix offers a concise and powerful way to perform rotations algebraically, enabling various applications across science and engineering fields.

Practical Tips for Solving Similar Exercises

- Always identify the rotation angle θ first.
- Remember the effects on the basis vectors to construct the matrix.
- Use known trigonometric identities to simplify calculations.
- Confirm properties such as orthogonality and determinant to verify the matrix's correctness.
- Be mindful of radians versus degrees when working with trigonometric functions.

Conclusion

Understanding how to find the standard matrix for a counterclockwise rotation in R^2 is a vital skill in linear algebra. It bridges geometric intuition with algebraic computation, providing a foundation for more complex transformations and applications. With the formula and method outlined above, students can confidently approach similar exercises, facilitate their understanding of linear transformations, and appreciate the elegance and utility of matrix representations in geometry.

Note: For exercises involving specific angles, always convert degrees to radians if necessary (since most trigonometric functions operate in radians). For example, 45° = $\pi/4$ radians.

Frequently Asked Questions

What is the standard matrix of a linear transformation in R^2 that rotates vectors counterclockwise by a certain angle?

The standard matrix for a counterclockwise rotation by angle θ in R^2 is [[cos θ, -sin θ], [sin θ, cos θ]].

How do you derive the matrix for a counterclockwise rotation in R^2?

You use the rotation formulas: for a vector (x, y), the rotated vector is (x cos θ - y sin θ, x sin θ + y cos θ), which leads to the matrix [[cos θ, -sin θ], [sin θ, cos θ]].

What are the key properties of the rotation matrix in R^2?

The rotation matrix is orthogonal with determinant 1, representing a pure rotation without scaling or shearing.

How does the angle of rotation affect the entries of the standard matrix?

The entries are functions of cos θ and sin θ, which depend on the rotation angle θ; as θ varies, the matrix changes accordingly.

Can the standard matrix of a counterclockwise rotation be used for reflections or other transformations?

No, the standard rotation matrix specifically represents rotation; reflections or other transformations have different matrices.

What is the effect of the rotation matrix on basis vectors in R^2?

Applying the rotation matrix to the basis vectors (1,0) and (0,1) rotates them by the specified angle, producing new basis directions.

How do you find the inverse of the rotation matrix in R^2?

The inverse of a rotation matrix is its transpose, which corresponds to rotation by the negative of the original angle: [[cos θ, sin θ], [-sin θ, cos θ]].

If a linear transformation rotates vectors counterclockwise by 45 degrees, what is its standard matrix?

The matrix is [[cos 45°, -sin 45°], [sin 45°, cos 45°]] which simplifies to [[▯ 2/2, -▯ 2/2], [▯ 2/2, ▯ 2/2]].

How do you verify that the standard matrix of a rotation preserves vector lengths and angles?

By confirming that the matrix is orthogonal with determinant 1, which ensures it preserves lengths and angles during rotation.

What is the significance of the standard matrix in understanding linear transformations in R^2?

The standard matrix provides a concrete representation of the transformation, allowing easy computation and analysis of its effects on vectors.

Additional Resources

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Introduction: Understanding Linear Transformations and Their Matrices

In the realm of linear algebra, the concept of linear transformations serves as a foundational pillar. These transformations facilitate the mapping of vectors from one vector space to another while preserving the operations of vector addition and scalar multiplication. When dealing with transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$, understanding how to determine their matrix representations—specifically, the standard matrices—is crucial for both theoretical exploration and practical application.

The exercises labeled 20-25 involve identifying the standard matrix associated with particular linear transformations, with one of these transformations being a rotation counterclockwise. Recognizing the nature of these transformations enables us to translate geometric operations into algebraic matrices, thus providing a powerful toolset for analysis and computation.

This article aims to thoroughly analyze the process of finding the standard matrix of a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, with a particular focus on the counterclockwise rotation. We will explore the fundamental principles, step-by-step procedures, and interpretative insights necessary for mastering this concept.

Linear Transformations in $\mathbb{R}^2$: Definitions and Properties

What is a Linear Transformation?

A linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a function that satisfies two key properties for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ and all scalars $c \in \mathbb{R}$:

1. Additivity: $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. Homogeneity: $T(c\mathbf{u}) = cT(\mathbf{u})$

These properties ensure that the transformation preserves the underlying vector space structure, enabling the representation of the transformation via matrices.

The Standard Matrix of a Linear Transformation

Any linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ can be represented by a 2×2 matrix A, known as the standard matrix. If $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ are the standard basis vectors in $\mathbb{R}^2$, then:

$$
A = \begin{bmatrix}
T(\mathbf{e}_1) & T(\mathbf{e}_2)
\end{bmatrix}
$$

This matrix acts on any vector $\mathbf{v} = (x, y)$ via matrix multiplication:

$$
T(\mathbf{v}) = A \mathbf{v}
$$

Therefore, understanding the images of the basis vectors under the transformation is key to constructing the matrix.

Rotations in $\mathbb{R}^2$: Geometric and Algebraic Perspectives

Geometric Intuition Behind Rotation

Rotation transformations are fundamental in geometry, describing the turning of vectors around a fixed point—commonly the origin. In $\mathbb{R}^2$, a counterclockwise rotation by an angle θ has specific geometric properties:

- It preserves the length (magnitude) of vectors.
- It preserves the angles between vectors.
- It maps each point to a new position obtained by rotating the original point around the origin.

These properties make rotations orthogonal transformations, which are isometries preserving distances and angles.

Algebraic Representation of Rotation

Mathematically, a counterclockwise rotation by an angle θ can be represented by the matrix:

$$
R_\theta = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
$$

This matrix acts on any vector $\mathbf{v} = (x, y)$ to produce a rotated vector $R_\theta \mathbf{v}$.

Deriving the Standard Matrix for Counterclockwise Rotation

Step 1: Understand the Transformation's Effect on Basis Vectors

Since the standard matrix is constructed from the images of the basis vectors $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$, the key is to determine:

- $T(\mathbf{e}_1)$
- $T(\mathbf{e}_2)$

For a counterclockwise rotation by θ:

$$
T(\mathbf{e}_1) = R_\theta \mathbf{e}_1 = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix}
\cos \theta \\
\sin \theta
\end{bmatrix}
$$

Similarly,

$$
T(\mathbf{e}_2) = R_\theta \mathbf{e}_2 = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix}
- \sin \theta \\
\cos \theta
\end{bmatrix}
$$

Step 2: Construct the Standard Matrix

Using the images of the basis vectors, the standard matrix A for the rotation transformation is:

$$
A = \begin{bmatrix}
T(\mathbf{e}_1) & T(\mathbf{e}_2)
\end{bmatrix} = \begin{bmatrix}
\cos \theta & - \sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
$$

This matrix directly encodes the rotation, and its application to any vector performs the counterclockwise rotation by θ.

Analytical Illustration: Practical Example

Suppose we are asked to find the standard matrix for a rotation by 90 degrees counterclockwise. Here, $\theta = 90^\circ = \pi/2$.

Step 1: Calculate $\cos \theta$ and $\sin \theta$:

$$
\cos \frac{\pi}{2} = 0, \quad \sin \frac{\pi}{2} = 1
$$

Step 2: Write the rotation matrix:

$$
R_{90^\circ} = \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
$$

Step 3: Verify the transformation of basis vectors:

- $T(\mathbf{e}_1) = (0, 1)$
- $T(\mathbf{e}_2) = (-1, 0)$

Step 4: Application:

Applying the matrix to a vector $\mathbf{v} = (x, y)$:

$$
R_{90^\circ} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix}
-y \\
x
\end{bmatrix}
$$

which geometrically confirms a 90-degree counterclockwise rotation.

Interpretation and Significance of the Rotation Matrix

Orthogonality and Determinant

The rotation matrix R_θ has several notable properties:

- Orthogonality: $R_\theta^T R_\theta = I$, meaning the transpose is equal to the inverse—a hallmark of orthogonal matrices.
- Determinant: $\det R_\theta = \cos^2 \theta + \sin^2 \theta = 1$, indicating an area-preserving transformation.

These properties confirm that a rotation is an isometry with no distortion, merely repositioning vectors without stretching or compressing.

Eigenvalues and Eigenvectors

The eigenvalues of the rotation matrix are complex unless $\theta = 0^\circ$ or 180°:

$$
\lambda = e^{i\theta}, \quad e^{-i\theta}
$$

which reflect the fact that vectors are rotated in the plane, with no real eigenvectors corresponding to the rotation unless \(\theta = 0^\circ\) or \(180^\circ\).

Applications in Engineering and Physics

Understanding the standard matrix of rotation is fundamental in fields like robotics, computer graphics, physics, and aerospace engineering, where precise control over orientations and movements is essential.

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