In Problems 2334, Find The Integrating Factor, The General Solu- Tion, And The Particular Solution Satisfying

In Problems 2334, Find The Integrating Factor, The General Solu- Tion, And The Particular Solution Satisfying a differential equation, understanding the process of solving first-order linear differential equations is essential. These problems often appear in calculus and differential equations courses, and mastering their solution methods is key to progressing in the subject. This article provides a comprehensive guide to identifying integrating factors, deriving the general solution, and finding particular solutions that satisfy given initial conditions, with examples and step-by-step instructions.

Understanding First-Order Linear Differential Equations

Before diving into specific problems, it is important to understand the general form of a first-order linear differential equation. These equations are typically written as:

$$ \frac{dy}{dx} + P(x) y = Q(x) $$

where $P(x)$ and $Q(x)$ are functions of x. The goal is to find a function $y(x)$ that satisfies this equation.

Finding the Integrating Factor

One of the key steps in solving first-order linear differential equations is determining the integrating factor $\mu(x)$. This factor simplifies the equation into an exact derivative, allowing us to integrate both sides easily.

Definition of the Integrating Factor

The integrating factor is given by:

$$ \mu(x) = e^{\int P(x) dx} $$

This exponential function, when multiplied through the original differential equation, transforms it into an integrable form.

Steps to Find the Integrating Factor

1. Identify $P(x)$: From the differential equation $\frac{dy}{dx} + P(x) y = Q(x)$.

2. Compute $\int P(x) dx$: Integrate the function $P(x)$ with respect to x.

3. Calculate $\mu(x) = e^{\int P(x) dx}$: Exponentiate the integral to find the integrating factor.

4. Verify the process: Multiply the entire differential equation by $\mu(x)$ to confirm that the left side becomes an exact derivative.

Example

Solve the differential equation:

$$\frac{dy}{dx} + 3 y = 6 x$$

- Identify $P(x) = 3$.

- Compute $\int 3 dx = 3x$.

- Find the integrating factor:

$$\mu(x) = e^{3x}$$

- Multiply through by e^{3x}:

$$e^{3x} \frac{dy}{dx} + 3 e^{3x} y = 6 x e^{3x}$$

which simplifies to:

$$\frac{d}{dx} \left(e^{3x} y \right) = 6 x e^{3x}$$

Deriving the General Solution

Once the integrating factor is identified, the next step is to find the general solution.

Integrating the Exact Derivative

The transformed differential equation becomes:

$$\frac{d}{dx} \left(\mu(x) y \right) = \mu(x) Q(x)$$

Integrate both sides with respect to x:

$$\mu(x) y = \int \mu(x) Q(x) dx + C$$

where C is an arbitrary constant representing the family of solutions.

Constructing the General Solution

Solve for y:

$$y = \frac{1}{\mu(x)} \left(\int \mu(x) Q(x) dx + C \right)$$

This expression provides the general solution to the differential equation.

Example Continued

Using the previous example:

$$e^{3x} y = \int 6 x e^{3x} dx + C$$

We now focus on computing:

$$\int 6 x e^{3x} dx$$

This integral can be approached with integration by parts:

- Let $u = 6x \Rightarrow du = 6 dx$

- Let $dv = e^{3x} dx \Rightarrow v = \frac{1}{3} e^{3x}$

Applying integration by parts:

$$\int 6 x e^{3x} dx = 6 x \times \frac{1}{3} e^{3x} - \int \frac{1}{3} e^{3x} \times 6 dx$$

$$= 2 x e^{3x} - 2 \int e^{3x} dx$$

$$= 2 x e^{3x} - 2 \times \frac{1}{3} e^{3x} + C'$$

$$= 2 x e^{3x} - \frac{2}{3} e^{3x} + C'$$

Substituting back:

$$e^{3x} y = 2 x e^{3x} - \frac{2}{3} e^{3x} + C$$

Dividing through by e^{3x}:

$$y = 2 x - \frac{2}{3} + C e^{-3x}$$

This is the general solution.

Finding Particular Solutions Satisfying Initial Conditions

In many applications, we are given an initial condition $y(x_0) = y_0$ and need to find a particular solution.

Procedure to Find the Particular Solution

1. Solve the differential equation to get the general solution.

2. Substitute the initial condition into the general solution:

$$y(x_0) = y_0$$

3. Solve for the constant C:

$$y_0 = \frac{1}{\mu(x_0)} \left(\int \mu(x) Q(x) dx \Big|_{x=x_0} + C \right)$$

4. Write the particular solution:

$$y_{part}(x) = \frac{1}{\mu(x)} \left(\int \mu(x) Q(x) dx + C_{value} \right)$$

Example with Initial Condition

Suppose the previous example has initial condition:

$$y(0) = 1$$

Using the general solution:

$$y = 2 x - \frac{2}{3} + C e^{-3x}$$

At $x=0$:

$$1 = 2 \times 0 - \frac{2}{3} + C e^{0}$$

$$1 = - \frac{2}{3} + C$$

$$C = 1 + \frac{2}{3} = \frac{5}{3}$$

Thus, the particular solution satisfying the initial condition is:

$$ y = 2 x - \frac{2}{3} + \frac{5}{3} e^{-3x} $$

Summary of the Solution Process

To solve first-order linear differential equations, follow these steps:

1. Identify $P(x)$ and $Q(x)$ in the equation $\frac{dy}{dx} + P(x) y = Q(x)$.
2. Calculate the integrating factor $\mu(x) = e^{\int P(x) dx}$.
3. Multiply the differential equation by $\mu(x)$ to obtain an exact derivative.
4. Integrate both sides to find $\mu(x) y$.
5. Solve for y to get the general solution.
6. If initial conditions are provided, substitute to find the particular solution.

Additional Tips and Common Mistakes

- Always check the form of the differential equation to confirm it is linear before proceeding.

- Be careful with the integration when calculating $\int P(x) dx$; use substitution or parts as needed.

- Remember that the integrating factor is always positive, as it's an exponential function.

- When integrating $\mu(x) Q(x)$, consider whether integration by parts or substitution simplifies the process.

- When applying initial conditions, ensure the point x_0 and y_0 are correctly substituted into the particular solution.

Conclusion

Problems like "Find The Integrating Factor, The General Solution, And The Particular Solution Satisfying" are foundational in solving first-order linear differential equations. By systematically identifying the integrating factor, transforming the differential equation into an exact form, and integrating, you can derive the general solution. Further, applying initial conditions allows you to find particular solutions tailored to specific scenarios. Mastery of these techniques provides a strong foundation for tackling more complex differential equations encountered in advanced mathematics, physics, engineering, and related fields.

Frequently Asked Questions

What is the first step in finding the integrating factor for a given differential equation?

The first step is to identify whether the differential equation is linear and in the form $dy/dx + P(x)y = Q(x)$. Then, find the integrating factor $\mu(x) = e^{\int P(x)dx}$.

How do you determine the general solution of a differential equation once the integrating factor is found?

Multiply the entire differential equation by the integrating factor to make it exact, then integrate both sides to find the general solution, which includes an arbitrary constant.

What is the process to find the particular solution satisfying initial conditions in Problem 2334?

After obtaining the general solution, substitute the initial conditions into the solution to solve for the constant, thus finding the particular solution that satisfies the given conditions.

Why is it important to verify that the integrating factor makes the differential equation exact?

Because an exact differential equation allows straightforward integration, ensuring that the method of integrating factors simplifies solving the differential equation correctly.

Can the integrating factor be a function of both x and y, and how does that affect solving the problem?

Yes, for some equations, the integrating factor depends on both x and y, but such cases are more complex and often require special methods beyond the standard integrating factor approach for linear equations.

What common mistakes should be avoided when solving Problems 2334 involving integrating factors and solutions?

Common mistakes include forgetting to determine if the equation is linear, incorrectly computing the integral for the integrating factor, and neglecting to apply initial conditions to find the particular solution.

Are there alternative methods to find solutions if the integrating factor approach is difficult or not applicable?

Yes, alternative methods include separation of variables, substitution methods, or using numerical methods if the differential equation does not fit the linear form suitable for integrating factors.

Additional Resources

In Problems 2334, Find The Integrating Factor, The General Solution, And The Particular Solution Satisfying

When tackling first-order differential equations, a systematic approach is essential for solving them effectively. Problems like 2334, which ask for finding the integrating factor, the general solution, and the particular solution satisfying given conditions, are foundational exercises that deepen understanding of differential equations' structure and solution techniques. This comprehensive review explores each component thoroughly, providing clarity and guidance for mastering such problems.

Understanding the Structure of First-Order Differential Equations

Before diving into the solution process, it is crucial to understand the types of first-order differential equations and their typical forms.

Standard Forms

- Linear Differential Equations: Of the form
$$
\frac{dy}{dx} + P(x) y = Q(x)
$$
where $P(x)$ and $Q(x)$ are continuous functions on an interval.

- Separable Equations: Can be written as
$$
\frac{dy}{dx} = g(x) h(y)
$$
which can be separated into integrals involving y and x.

- Exact Equations: Have the form
$$
M(x,y) + N(x,y) \frac{dy}{dx} = 0
$$
where $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

- Non-Exact Equations: Can sometimes be made exact through an integrating factor.

Most problems like 2334 involve linear or non-exact equations requiring an integrating factor.

Finding the Integrating Factor

The integrating factor is a pivotal concept that transforms a non-exact differential equation into an exact one, enabling straightforward integration.

What Is an Integrating Factor?

An integrating factor $\mu(x)$ (or $\mu(y)$, or $\mu(x,y)$ in more complex cases) is a function multiplied throughout an equation to make it exact.

For a linear differential equation:
$$
\frac{dy}{dx} + P(x) y = Q(x)
$$
the integrating factor is typically:
$$
\mu(x) = e^{\int P(x) dx}
$$
This choice ensures the left side becomes the derivative of $\mu(x) y$.

Steps to Find the Integrating Factor

1. Identify the form of the differential equation.
Confirm whether it is linear or can be transformed into a linear form.

2. Compute the integral of $P(x)$.
For the standard form, this is straightforward:
$$
\int P(x) dx
$$

3. Calculate the integrating factor $\mu(x)$.
Use the exponential of the integral:
$$
\mu(x) = e^{\int P(x) dx}
$$

\]

4. Multiply the entire differential equation by $\mu(x)$.
This converts the equation into an exact differential form.

Example:
Suppose the differential equation is:
$$
\frac{dy}{dx} + \frac{2}{x} y = \sin x
$$
- $P(x) = \frac{2}{x}$
- Integrate:
$$
\int \frac{2}{x} dx = 2 \ln |x| \Rightarrow \mu(x) = e^{2 \ln |x|} = |x|^2
$$
- Multiply through by $\mu(x)$:
$$
x^2 \frac{dy}{dx} + 2x y = x^2 \sin x
$$
- Recognize the left side as the derivative of $x^2 y$:
$$
\frac{d}{dx} (x^2 y) = x^2 \sin x
$$

Obtaining the General Solution

Once the differential equation is made exact (either inherently or via an integrating factor), the next step is to find the general solution.

Methodology for the General Solution

1. Express the exact differential as a total differential:
$$
dF(x,y) = 0
$$
where $F(x,y)$ is an potential function satisfying:
$$
\frac{\partial F}{\partial x} = M(x,y), \quad \frac{\partial F}{\partial y} = N(x,y)
$$

2. Integrate $M(x,y)$ with respect to x:
$$
F(x,y) = \int M(x,y) dx + h(y)
$$
where $h(y)$ is an arbitrary function of y.

3. Differentiate $F(x,y)$ with respect to y:

$$
\frac{\partial F}{\partial y} = N(x,y)
$$

Use this to determine $h(y)$.

4. Form the general solution:

$$
F(x,y) = C
$$

where C is an arbitrary constant.

Explicit Solution Examples

Continuing from the previous example:

$$
\frac{d}{dx} (x^2 y) = x^2 \sin x
$$

- Integrate both sides:

$$
x^2 y = \int x^2 \sin x \, dx + C
$$

- The integral $\int x^2 \sin x dx$ can be solved via integration by parts multiple times or by tabular integration.

Integration by parts:
Let:
- $u = x^2 \Rightarrow du = 2x dx$
- $dv = \sin x dx \Rightarrow v = -\cos x$

Then:

$$
\int x^2 \sin x dx = -x^2 \cos x + 2 \int x \cos x dx
$$

Next integral:
- $u = x \Rightarrow du = dx$
- $dv = \cos x dx \Rightarrow v = \sin x$

So:

$$
\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + C
$$

Putting it all together:

$$
\int x^2 \sin x dx = -x^2 \cos x + 2 (x \sin x + \cos x) + C
$$

Therefore:

$$
x^2 y = -x^2 \cos x + 2 x \sin x + 2 \cos x + C
$$

and the general solution:

$$
y = \frac{-x^2 \cos x + 2 x \sin x + 2 \cos x + C}{x^2}
$$

Finding the Particular Solution Satisfying Conditions

The general solution contains an arbitrary constant C. To find the particular solution that satisfies specific initial or boundary conditions, follow these steps:

Applying Initial or Boundary Conditions

Suppose the problem states:

$$
y(x_0) = y_0
$$

then:

$$
y_0 = \frac{-x_0^2 \cos x_0 + 2 x_0 \sin x_0 + 2 \cos x_0 + C}{x_0^2}
$$

Solve for C:

$$
C = y_0 x_0^2 + x_0^2 \cos x_0 - 2 x_0 \sin x_0 - 2 \cos x_0
$$

Substitute back into the general solution:

$$
y = \frac{-x^2 \cos x + 2 x \sin x + 2 \cos x + C}{x^2}
$$

which yields the particular solution.

Handling Non-Exact Equations Without Immediate Integrating Factors

Some differential equations are not directly exact and do not have obvious integrating factors. In such cases, the approach involves:

Finding an Integrating Factor Dependent on \(x \) or \(y \)

- Compare:
\[
\frac{\partial M}{\partial y} \quad \text{and} \quad \frac{\partial N}{\partial x}
\]
and determine whether an integrating factor exists as a function solely of \(x \) or \(y \).

- If \(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \) is divisible by a function of \(x \) or \(y \), an integrating factor can sometimes be constructed.

Using the Formula for an Integrating Factor

- For an equation \(M(x,y) + N(x,y) \frac{dy}{dx} = 0 \), if:
\[
\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \text{

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