Is The Function F(z) = 1 (1z) 2 Complex Differentiable At Z = 0? If Yes, Then Find Its Power Series Expansion

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Understanding whether a complex function is differentiable at a certain point is fundamental in complex analysis. When analyzing functions like F(z) = 1 / (1z)^2, it's essential to explore their differentiability at specific points, such as z = 0. In this article, we will examine if the function F(z) = 1 / (1z)^2 is complex differentiable at z = 0, and if it is, how to find its power series expansion around that point. This exploration involves delving into concepts like complex differentiability, Laurent series, and the nature of singularities.

Analyzing the Function F(z) = 1 / (1z)^2

Before diving into differentiability, it's crucial to understand the form and domain of the function.

Understanding the Function's Definition

The given function is F(z) = 1 / (1z)^2. This notation appears to be a typographical error or an unconventional representation. Typically, functions involving such notation are written as:

- F(z) = 1 / (1 - z)^2

Assuming the intended function is F(z) = 1 / (1 - z)^2, which is a common function studied in complex analysis, especially in the context of power series expansion around z = 0.

If the function is indeed F(z) = 1 / (1 - z)^2, then:

- It is a rational function with a potential singularity at z = 1.
- Its behavior at z = 0, which is within its domain, can be analyzed for differentiability and power series expansion.

Note: If the original function differs, please clarify. For this article, we'll proceed with F(z) = 1 / (1 - z)^2.

Is F(z) = 1 / (1 - z)^2 Complex Differentiable at z = 0?

Complex differentiability at a point involves the existence of the complex derivative at that point. For rational functions like F(z), differentiability depends on whether the function is analytic (holomorphic) at the point in question.

Domain of F(z)

- $F(z) = 1 / (1 - z)^2$ is defined everywhere except at $z = 1$, where it has a singularity.
- Since $z = 0 \neq 1$, $F(z)$ is defined at $z = 0$.

Is F(z) Holomorphic at z = 0?

- Because $F(z)$ is a rational function with a pole only at $z = 1$, it is holomorphic (complex differentiable) at $z = 0$.
- Therefore, $F(z)$ is complex differentiable at $z = 0$.

Conclusion: Yes, $F(z) = 1 / (1 - z)^2$ is complex differentiable at $z = 0$.

Finding the Power Series Expansion of F(z) Around z = 0

Having established that $F(z)$ is differentiable at $z = 0$, the next step is to derive its power series expansion (Taylor series) centered at $z = 0$.

Expressing F(z) as a Power Series

- Recall that the geometric series expansion for $|z| < 1$ is:

$$
\frac{1}{1 - z} = \sum_{n=0}^\infty z^n
$$

- To find the expansion for $F(z) = 1 / (1 - z)^2$, differentiate the geometric series:

$$
\frac{d}{dz} \left(\frac{1}{1 - z} \right) = \frac{1}{(1 - z)^2}
$$

- Differentiating term-by-term:

$$
\frac{d}{dz} \left(\sum_{n=0}^\infty z^n \right) = \sum_{n=1}^\infty n z^{n-1}
$$

- Therefore:

$$
\frac{1}{(1 - z)^2} = \sum_{n=1}^\infty n z^{n-1}
$$

- Reindexing the sum:

$$
\frac{1}{(1 - z)^2} = \sum_{n=0}^\infty (n+1) z^n
$$

for $|z| < 1$.

Result: The power series expansion of $F(z) = 1 / (1 - z)^2$ around $z = 0$ is:

$$
F(z) = \sum_{n=0}^\infty (n+1) z^n
$$

Summary of the Power Series Expansion

- The expansion converges for $|z| < 1$.
- The coefficients are $(n+1)$, starting from $n=0$.

Analysis of Singularity and Radius of Convergence

Understanding the nature of the singularity is essential for a complete picture.

Type of Singularity at z = 1

- The point $z = 1$ is a pole of order 2 for $F(z)$.
- The Laurent series expansion around $z = 1$ would involve negative powers, indicating a pole.

Radius of Convergence

- The power series expansion centered at $z = 0$ has a radius of convergence of 1, limited by the nearest singularity at $z = 1$.

Summary of Key Points

- The function $F(z) = 1 / (1 - z)^2$ is complex differentiable at $z = 0$ because it is holomorphic everywhere except at $z = 1$.

- The power series expansion around $z = 0$ is given by $\sum_{n=0}^\infty (n+1) z^n$, valid for $|z| < 1$.

- The singularity at $z = 1$ is a pole of order 2, which determines the radius of convergence of the power series.

Additional Insights and Applications

Using the Power Series for Computations

- The expansion allows for easy computation of $F(z)$ near $z = 0$.
- It provides insights into the behavior of the function, especially for small $|z|$.

Implications in Complex Analysis

- The derivation exemplifies how geometric series and differentiation can be used to find power series expansions.
- Understanding singularities helps in contour integration and residue calculations.

Conclusion

In conclusion, assuming the function in question is $F(z) = 1 / (1 - z)^2$, it is indeed complex differentiable at $z = 0$. Its power series expansion around $z = 0$ is:

$$
F(z) = \sum_{n=0}^\infty (n+1) z^n
$$

valid within the radius $|z| < 1$. This expansion is invaluable for analyzing the function's behavior, performing approximations, and understanding its singularities. If the original function was different or if additional context is provided, similar analysis can be carried out to determine differentiability and power series representations accordingly.

Frequently Asked Questions

Is the function F(z) = 1 / (1z)^2 complex differentiable at z = 0?

No, the function $F(z) = 1 / (1z)^2$ is not complex differentiable at $z = 0$ because it has a pole there, and the function is not analytic at that point.

What is the domain of the function F(z) = 1 / (1z)^2?

The domain of $F(z) = 1 / (1z)^2$ is all complex numbers z except $z = 0$, where the function has a singularity.

Can F(z) = 1 / (1z)^2 be expanded into a power series around z = 0?

No, since F(z) has a pole at z = 0, it cannot be expressed as a power series (which requires analyticity) around that point.

Is F(z) = 1 / (1z)^2 an entire function?

No, F(z) = 1 / (1z)^2 is not entire because it is not analytic at z = 0 due to the singularity.

How do you determine whether F(z) is complex differentiable at a point?

You examine whether F(z) is analytic at that point; if it is differentiable in some neighborhood, then it is complex differentiable there. For F(z) at z=0, the singularity prevents differentiability.

What is the Laurent series expansion of F(z) around z = 0?

F(z) = 1 / (z)^2, which is a Laurent series with only a principal part: F(z) = z^{-2}, indicating a pole at z=0.

If F(z) is not differentiable at z=0, can we find its power series expansion there?

No, because a power series expansion requires the function to be analytic at that point, which is not the case here due to the pole at z=0.

What is the significance of the pole at z=0 for the function F(z)?

The pole indicates that F(z) approaches infinity as z approaches 0, and this singularity prevents the function from being complex differentiable at that point.

How would the behavior of F(z) change if the function were modified to F(z) = 1 / (z)^2?

F(z) = 1 / (z)^2 is similar to the original function but explicitly shows the pole at z=0, confirming that the function is not differentiable there and has a Laurent series expansion starting with z^{-2}.

Additional Resources

Is The Function F(z) = 1 / $(1z)^2$ Complex Differentiable At Z = 0? If Yes, Then Find Its Power Series Expansion

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Introduction

The question of complex differentiability is fundamental in the field of complex analysis, underpinning many of its core principles and applications. Specifically, understanding whether a given function is complex differentiable at a point—and if so, how to express it via a power series expansion—provides deep insight into its analytical properties. This article rigorously investigates the function $F(z) = \frac{1}{(1z)^2}$ at $z = 0$, exploring its differentiability status and, where applicable, deriving its power series expansion.

Clarifying the Function $F(z)$

Before delving into the analysis, it’s essential to interpret the given function accurately. The notation:

$$F(z) = \frac{1}{(1z)^2}$$

might initially seem ambiguous due to the expression $1z$. Typically, in complex analysis, such notation might represent:

1. A typo or misprint, intending to write $(1 + z)$ instead of $1z$.
2. An explicit product, i.e., $1 \times z$.

Given common conventions and standard functions studied in complex analysis, the most plausible and meaningful interpretation is:

$$F(z) = \frac{1}{(1 + z)^2}$$

This form aligns with a well-known class of functions whose differentiability and series expansions are extensively analyzed.

Therefore, the focus of this investigation is on

$$\boxed{F(z) = \frac{1}{(1 + z)^2}}$$

Part 1: Complex Differentiability at $z=0$

Analyticity and Differentiability in Complex Analysis

A function $F(z)$ is said to be complex differentiable at a point z_0 if the limit

$$F'(z_0) = \lim_{h \to 0} \frac{F(z_0 + h) - F(z_0)}{h}$$

exists and is finite.

A function that is complex differentiable in an open neighborhood around z_0 is called holomorphic or analytic at z_0.

Domain of $F(z) = \frac{1}{(1 + z)^2}$

The function $F(z)$ is a rational function with a potential singularity where its denominator vanishes:

$$
(1 + z)^2 = 0 \implies z = -1
$$

This is a pole of order 2 at $z = -1$.

- At $z = 0$, the denominator $(1 + 0)^2 = 1 \neq 0$, so $F(z)$ is well-defined and finite at zero.

Is $F(z)$ differentiable at $z=0$?

Since $F(z)$ is rational with no singularity at $z=0$, it is holomorphic (complex differentiable) in some neighborhood of $z=0$. Specifically:

- $F(z)$ is analytic everywhere except at $z=-1$.

- At $z=0$, which is away from the pole, $F(z)$ is complex differentiable.

Conclusion:

Yes, $F(z) = \frac{1}{(1 + z)^2}$ is complex differentiable at $z=0$.

Part 2: Deriving the Power Series Expansion of $F(z)$ Around $z=0$

The Approach

Expressing $F(z)$ as a power series expansion around $z=0$ involves representing it in terms of a geometric series or binomial series, depending on the form.

Step 1: Recognize the relation to geometric series

Note that:

$$
F(z) = \frac{1}{(1 + z)^2}
$$

which resembles the derivative of a geometric series. Recall:

$$
\frac{1}{1 + z} = \sum_{n=0}^\infty (-1)^n z^n, \quad \text{for } |z| < 1
$$

Differentiating both sides:

$$
\frac{d}{dz} \left(\frac{1}{1 + z}\right) = -\frac{1}{(1 + z)^2} = \sum_{n=1}^\infty (-1)^n n z^{n-1}
$$

Thus,

$$
\frac{1}{(1 + z)^2} = - \sum_{n=1}^\infty (-1)^n n z^{n-1}
$$

or equivalently,

$$
F(z) = \frac{1}{(1 + z)^2} = \sum_{n=1}^\infty n (-1)^{n+1} z^{n-1}
$$

Step 2: Write the series explicitly

Expressed as a power series centered at $z=0$:

$$
\boxed{
F(z) = \sum_{n=1}^\infty n (-1)^{n+1} z^{n-1}
}
$$

which converges for $|z| < 1$.

Step 3: Alternative form

Re-indexing to write in standard form:

$$
F(z) = \sum_{k=0}^\infty (k+1)(-1)^{k+2} z^{k}
$$

Since $(-1)^{k+2} = (-1)^k$, the series simplifies to:

$$
F(z) = \sum_{k=0}^\infty (k+1) (-1)^k z^{k}
$$

Final power series expansion:

$$
\boxed{
F(z) = \sum_{k=0}^\infty (k+1) (-1)^k z^{k} \quad \text{for } |z| < 1
}
$$

Part 3: Validity and Radius of Convergence

The derived series originates from the geometric series expansion, which converges absolutely for $|z| < 1$.

- The function $F(z)$ is analytic on the disk $|z| < 1$, excluding the singularity at $z=-1$.

- The power series provides a valid local expansion at $z=0$, with radius of convergence $R=1$.

Part 4: Broader Implications and Applications

Analytic Continuation

While the power series converges only in $|z| < 1$, the function $F(z)$ remains well-defined and differentiable everywhere except at $z=-1$. Analytic continuation extends the understanding of $F(z)$ beyond the disk, but the power series expansion’s radius is limited by the nearest singularity.

Significance in Complex Analysis

The explicit derivation of the power series expansion demonstrates the utility of geometric series and differentiation techniques in analyzing complex functions. This approach is fundamental in many branches of complex analysis, including residue calculus, Laurent series, and the study of singularities.

Conclusion

The thorough investigation confirms that:

- Yes, the function $F(z) = \frac{1}{(1 + z)^2}$ is complex differentiable at $z=0$.

- Its power series expansion around $z=0$ is:

$$
\boxed{
F(z) = \sum_{k=0}^\infty (k+1)(-1)^k z^{k}, \quad |z| < 1
}
$$

This series provides a powerful local representation, capturing the behavior of $F(z)$ within its radius of convergence. Understanding such expansions is vital for deeper insights into the nature of rational functions and their singularities in complex analysis.

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