Set Up Only An Iterated Double Integral That Gives The Volume Of The Solid Enclosed Above By The Hyperboloid

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Set Up Only An Iterated Double Integral That Gives The Volume Of The Solid Enclosed Above By The Hyperboloid is a fundamental problem in multivariable calculus, involving the calculation of volumes bounded by complex surfaces. Hyperboloids are quadratic surfaces that appear frequently in geometry, physics, and engineering. When tasked with determining the volume of the solid enclosed above a hyperboloid, the key step is to properly set up an iterated double integral that accurately reflects the geometry of the surface and the region of integration. This article will guide you through the process, discussing the hyperboloid's equations, the determination of the projection in the xy-plane, and the systematic setup of the integral for volume calculation.

Understanding the Hyperboloid and Its Equation

Types of Hyperboloids

- **One-sheet hyperboloid:** An equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.
- **Two-sheet hyperboloid:** An equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$.

Typically, the problem involves the one-sheet hyperboloid, which opens along the z-axis, or the two-sheet hyperboloid, which consists of two disconnected parts. For volume calculations above the hyperboloid, understanding the specific form is crucial.

Standard Equation of the Hyperboloid

A common form of the one-sheet hyperboloid is:

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1
$$

Rearranged to express z in terms of x and y, it becomes:

$$
z = \pm c \sqrt{ \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 }
$$

For the volume above the hyperboloid, focus on the upper sheet, where:

$$
z = c \sqrt{ \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 }
$$

Determining the Projection Region in the xy-Plane

Identifying the Region D

The projection of the solid onto the xy-plane, denoted as D, is critical in setting up the double integral. It represents the set of all points (x, y) for which the hyperboloid exists above the xy-plane.

Conditions for the Projection

- The expression under the square root must be non-negative:

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \geq 0
$$

- This simplifies to:

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1
$$

Geometric Description of the Projection

In the xy-plane, the region D is the exterior of the ellipse:

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
$$

Thus, the solid above the hyperboloid extends over the region where:

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1
$$

Setting Up the Double Integral for the Volume

Expressing the Volume as a Double Integral

The volume V of the solid enclosed above the hyperboloid can be expressed as:

$$
V = \iint_D \left[\text{height at each point} \right] \, dx\,dy
$$

Since the height at each point (x, y) is given by the upper sheet of the hyperboloid:

$$
z = c \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1}
$$

Formulating the Integral

The volume becomes:

$$
V = \iint_D c \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1} \, dx\,dy
$$

Choosing a Suitable Coordinate System

To simplify the integral, consider the symmetry of the region D. Since the region is outside an ellipse, it's often helpful to use elliptical coordinates or transform the variables to standard forms to facilitate integration.

Parameterizing the Region in Elliptical Coordinates

Transformations for Simplification

- Define new variables:

$$
u = \frac{x}{a}, \quad v = \frac{y}{b}
$$

- The Jacobian determinant for this change of variables is:

$$
|J| = a b
$$

Rewriting the Integral in Terms of u and v

$$
V = c a b \iint_{u^2 + v^2 \geq 1} \sqrt{u^2 + v^2 - 1} \, du \, dv
$$

Converting to Polar Coordinates for the Inner Region

Polar Coordinates in the (u,v)-Plane

- Set:

$$
u = r \cos \theta, \quad v = r \sin \theta
$$

- The Jacobian for the polar coordinate transformation is:

$$
|J_{polar}| = r
$$

Expressing the Volume Integral in Polar Coordinates

$$
V = c a b \int_{0}^{2\pi} \int_{1}^{\infty} \sqrt{r^2 - 1} \, r \, dr \, d\theta
$$

Final Setup of the Iterated Double Integral

Explicit Integral Expression

Combining all the transformations, the volume of the solid above the hyperboloid is given by:

$$
V = c a b \int_{0}^{2\pi} \int_{1}^{\infty} \sqrt{r^2 - 1} \, r \, dr \, d\theta
$$

Simplifying the Integral

1. Integrate with respect to r first:

$$
I = \int_{1}^{\infty} r \sqrt{r^2 - 1} \, dr
$$

2. Then multiply by the angular integral:

$$
V = 2 \pi c a b \times I
$$

Techniques for Evaluating the Integral

Substitution for the Radial Integral

- Let:

$$
t = r^2 - 1 \quad \Rightarrow \quad dt = 2 r \, dr
$$

- Express the integral I in terms of t:

$$
I = \frac{1}{2} \int_{0}^{\infty} \sqrt{t} \, dt
$$

Evaluating the Integral

- The integral $\int_{0}^{\infty} t^{1/2} dt$ diverges, indicating that the volume extends infinitely, which is typical for un

Frequently Asked Questions

How do I set up an iterated double integral to find the volume of the solid enclosed above by a hyperboloid?

First, identify the equation of the hyperboloid, determine its projection onto the xy-plane, and then express the limits for z in terms of x and y. The double integral is set up over the projection region, with the upper boundary given by the hyperboloid's equation, integrating the differential element dz dy dx or in an iterated form accordingly.

What are the steps to determine the limits of integration for the double integral over a hyperboloid?

Begin by analyzing the hyperboloid's equation to find the projection region in the xy-plane. Determine the boundary curves (such as circles or hyperbolas) that define this region. Then, for each point in this region, find the corresponding z-limits from the hyperboloid's equation, typically from the hyperboloid surface up to the xy-plane or the specified bounds.

Which coordinate system is most convenient for setting up the integral over a hyperboloid, Cartesian or cylindrical?

Cylindrical coordinates are often more convenient because hyperboloids are symmetric about an axis, simplifying the equations and limits. Converting to cylindrical coordinates reduces the integral to more manageable forms, especially when the hyperboloid's equation involves r and θ naturally.

Can you provide the general form of the double integral for the **volume above a hyperboloid z = f(x,y)?**

Yes, the general form of the double integral for the volume is: $V = \iint_D [z_{upper}(x,y) - z_{lower}(x,y)]\, dA$, where D is the projection region in the xy-plane, and z_upper and z_lower define the upper and lower surfaces (for a solid above a surface, z_lower could be the xy-plane or another boundary). For a hyperboloid, z_upper is given by the hyperboloid equation, and z_lower is often zero or another boundary.

What considerations are important when choosing the limits of integration for the hyperboloid volume problem?

Key considerations include accurately describing the projection region in the xy-plane, ensuring the limits reflect the hyperboloid's intersection with other surfaces or boundaries, and simplifying the integral by choosing coordinates that exploit symmetry. Additionally, verify that the limits correctly capture the entire volume without omission or overlap.

Additional Resources

Set Up Only An Iterated Double Integral That Gives The Volume Of The Solid Enclosed Above By The Hyperboloid

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Introduction

In the realm of multivariable calculus, the calculation of volume enclosed by surfaces is a foundational yet intricate endeavor. Among the various surfaces studied, the hyperboloid stands out due to its distinctive saddle-shaped or hourglass geometry, which presents both challenges and opportunities for integral setup. This article explores the process of establishing an iterated double integral that precisely calculates the volume of the solid enclosed above by a hyperboloid, emphasizing a systematic and rigorous approach suitable for advanced readers and practitioners.

The Hyperboloid: Definition and Geometric Properties

Types of Hyperboloids

Hyperboloids are quadratic surfaces characterized by equations of the form:

- One-sheet hyperboloid:

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1
$$

- Two-sheet hyperboloid:

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1
$$

This discussion focuses on the one-sheet hyperboloid, which forms a continuous surface extending infinitely along the z-axis, with the "waist" at the origin.

Geometric Features

- The hyperboloid opens upward and downward along the z-axis.
- Cross sections parallel to the xy-plane are ellipses:

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} = \text{constant}
$$

- The surface exhibits rotational symmetry if $a = b$, simplifying calculations.

Establishing the Region of Integration

Visualizing the Enclosed Solid

The problem asks for the volume enclosed above the hyperboloid surface, implying a bounded region from below by the hyperboloid itself and possibly limited by other surfaces or bounds at certain heights.

For the purpose of this investigation, assume we are considering the volume between a hyperboloid and a plane or within a finite region, such as:

- The volume above the hyperboloid surface $z = z(x,y)$ up to a certain plane $z = Z$.
- Or, the volume enclosed between the hyperboloid and a plane $z = Z$, with the hyperboloid extending infinitely, but the volume truncated at $z = Z$.

To establish a clear setup, consider the solid bounded above by the hyperboloid and below by the plane $z=0$.

Formulating the Volume as an Integral

The goal is to set up an iterated double integral that computes this volume. The key steps involve:

1. Expressing the hyperboloid explicitly as a function $z = z(x,y)$.
2. Determining the projection region D in the xy-plane over which the volume extends.
3. Integrating the height difference $z(x,y) - 0$ over the region D.

Expressing the Hyperboloid as a Function: $z = z(x,y)$

Starting from the hyperboloid equation:

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1
$$

Solve for z:

$$
z^2 = c^2 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right)
$$

Since we're interested in the volume above the hyperboloid, which is the region where $z \geq 0$, and assuming the hyperboloid opens upward, then:

$$
z(x,y) = c \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1}
$$

Note: The expression under the square root must be non-negative:

\[

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1
$$

This defines the projection region D.

Determining the Projection Region D

The projection D onto the xy-plane consists of all (x,y) satisfying:

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1
$$

which geometrically corresponds to the exterior of an ellipse.

However, if we are considering the volume inside the hyperboloid (bounded above), it is more typical to consider the interior of the hyperboloid, which occurs when:

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1
$$

and the hyperboloid is above the plane $z=0$.

Therefore, for the volume above the hyperboloid, the relevant projection is:

$$
D = \left\{ (x,y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}
$$

which is the interior of an ellipse in the xy-plane.

Setting Up the Double Integral

Given the above, the volume V can be expressed as:

$$
V = \iint_{D} z(x,y) \, dx\, dy
$$

where:

$$
z(x,y) = c \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}
$$

Note: The negative root is discarded because it would correspond to the region below the plane $z=0$.

Thus, the volume is:

$$
V = \iint_{D} c \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \, dx\, dy
$$

Choosing Coordinates for Integration

To evaluate or at least clearly set up this integral, it's advantageous to transform into a coordinate system aligned with the ellipse D.

Elliptical Coordinates:

Define:

$$
x = a r \cos \theta, \quad y = b r \sin \theta
$$

where:

- $r \in [0,1]$,
- $\theta \in [0, 2\pi]$.

The Jacobian determinant for this transformation is:

$$
J = \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| = a b r
$$

Thus, the differential area element:

$$
dx\, dy = a b r\, dr\, d\theta
$$

Expressing the Integral in Elliptical Coordinates

The integral becomes:

$$
V = \int_{0}^{2\pi} \int_{0}^{1} c \sqrt{1 - r^2} \times a b r\, dr\, d\theta
$$

which simplifies to:

$$
V = a b c \int_{0}^{2\pi} \left(\int_{0}^{1} r \sqrt{1 - r^2} \, dr \right) d\theta
$$

Since the integrand is independent of θ, the θ-integral is straightforward:

$$
V = 2\pi a b c \int_{0}^{1} r \sqrt{1 - r^2} \, dr
$$

Evaluating the Inner Integral

The remaining integral:

$$
I = \int_{0}^{1} r \sqrt{1 - r^2} \, dr
$$

can be evaluated via substitution:

Let $u = 1 - r^2$, then:

$$
du = -2 r\, dr \Rightarrow r\, dr = -\frac{1}{2} du
$$

When $r=0$, $u=1$; when $r=1$, $u=0$.

Thus,

$$
I = -\frac{1}{2} \int_{u=1}^{0} \sqrt{u}\, du = \frac{1}{2} \int_{0}^{1} u^{1/2} \, du
$$

which evaluates to:

$$
\frac{1}{2} \times \frac{u^{3/2}}{\frac{3}{2}} \bigg|_{0}^{1} = \frac{1}{2} \times \frac{2}{3} (1^{3/2} - 0) = \frac{1}{3}
$$

Final Expression for the Volume

Putting it all together:

$$
V = 2\pi a b c \times \frac{1}{3} = \frac{2\pi}{3} a b c
$$

Therefore, the volume enclosed above the hyperboloid (bounded within the elliptical projection in the xy-plane) is:

\[
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V = \frac{2\

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