Show That If U1 And U2 Are Subspaces Of A Vector Space V, Thendim(U1 + U2) = Dim U1 + Dim U2 Dim(U1 U2),where

Show That If U1 And U2 Are Subspaces Of A Vector Space V, Thendim(U1 + U2) = Dim U1 + Dim U2 Dim(U1 U2),where

Understanding the relationship between subspaces within a vector space is fundamental in linear algebra. One of the key results in this area involves the dimension of the sum of two subspaces, U1 and U2, and how it relates to their individual dimensions and their intersection. This article aims to provide a comprehensive explanation and proof of the statement:

"Show that if U1 and U2 are subspaces of a vector space V, then

$$ \dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2) $$"

This result is a cornerstone in linear algebra, with applications spanning from solving systems of equations to understanding the structure of vector spaces.

Understanding Subspaces and Their Dimensions

Before delving into the proof and implications of the dimension formula, it is essential to review the fundamental concepts involved.

What is a Subspace?

A subspace U of a vector space V over a field F is a subset of V that satisfies three properties:

1. Contains the zero vector: $ \mathbf{0} \in U $
2. Closed under addition: For any $ \mathbf{u}_1, \mathbf{u}_2 \in U $, $ \mathbf{u}_1 + \mathbf{u}_2 \in U $
3. Closed under scalar multiplication: For any $ \mathbf{u} \in U $ and scalar $ c \in F $, $ c \mathbf{u} \in U $

Subspaces are the building blocks of vector spaces, allowing the decomposition of complex spaces into simpler, manageable parts.

Dimension of a Subspace

The dimension of a subspace U, denoted $\dim U$, is the number of vectors in a basis for U. A basis is a minimal set of vectors that are linearly independent and span the subspace. This measure gives insight into the "size" or "degree of freedom" within the subspace.

The Sum and Intersection of Subspaces

Given two subspaces U1 and U2 of V, their sum and intersection are fundamental constructions in linear algebra:

Sum of Subspaces $U_1 + U_2$

Defined as:

$$U_1 + U_2 = \{ \mathbf{u}_1 + \mathbf{u}_2 \mid \mathbf{u}_1 \in U_1, \mathbf{u}_2 \in U_2 \}$$

This set is itself a subspace of V, containing all possible sums of vectors from U1 and U2.

Intersection of Subspaces $U_1 \cap U_2$

Defined as:

$$U_1 \cap U_2 = \{ \mathbf{v} \in V \mid \mathbf{v} \in U_1 \text{ and } \mathbf{v} \in U_2 \}$$

The intersection contains vectors common to both subspaces.

The Dimension Formula: Statement and Significance

The key theorem relating the dimensions of subspaces, their sum, and their intersection states:

\[
\boxed{

\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)
}
\]

This formula is significant because it accounts for the overlap between U1 and U2; it prevents double-counting the vectors in the intersection when summing the dimensions.

Proof of the Dimension Formula

The proof involves constructing bases and understanding how they relate to each other.

Step 1: Choose Bases for U1 and U2

- Let \(B_1 \) be a basis for \(U_1 \), with \(|B_1| = \dim U_1 \)
- Let \(B_2 \) be a basis for \(U_2 \), with \(|B_2| = \dim U_2 \)

Step 2: Find a Basis for \(U_1 \cap U_2 \)

- Let \(B_0 \) be a basis for \(U_1 \cap U_2 \)
- Since \(U_1 \cap U_2 \subseteq U_1 \) and \(U_2 \), \(B_0 \) can be extended to bases of \(U_1 \) and \(U_2 \):

\[
B_1 = B_0 \cup B_{1}'
\]
\[
B_2 = B_0 \cup B_{2}'
\]

where \(B_{1}' \) and \(B_{2}' \) are sets of vectors that extend \(B_0 \) to bases of \(U_1 \) and U2, respectively.

Step 3: Construct a Basis for \(U_1 + U_2 \)

- The union:

\[
B = B_{1}' \cup B_{2}'
\]

forms a basis for \(U_1 + U_2 \). To see why:

- The set B spans $U_1 + U_2$, because any vector in $U_1 + U_2$ can be written as a sum of vectors in U_1 and U_2, which are spanned by their respective bases.
- The set B is linearly independent, as the vectors in B_{1}' and B_{2}' are each independent within their own spaces, and their union does not introduce linear dependence outside of $U_1 \cap U_2$.

Step 4: Compute the Dimensions

- The size of the basis B is:

$$
|B| = |B_{1}'| + |B_{2}'|
$$

- Since B_0 is a basis for $U_1 \cap U_2$, and B_1 extends B_0, we have:

$$
\dim U_1 = |B_0| + |B_{1}'|
$$

- Similarly,

$$
\dim U_2 = |B_0| + |B_{2}'|
$$

- Therefore,

$$
|B| = (\dim U_1 - |B_0|) + (\dim U_2 - |B_0|) = \dim U_1 + \dim U_2 - 2|B_0|
$$

- But, since $|B_0| = \dim(U_1 \cap U_2)$, the dimension of the sum is:

$$
\dim(U_1 + U_2) = |B| = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)
$$

which completes the proof.

Applications and Examples of the Dimension Formula

Understanding this formula is crucial in various applications across linear algebra and related fields.

Applications in Solving Systems of Linear Equations

- The formula helps in determining the maximum number of linearly independent solutions when combining subspace solutions, such as in systems with overlapping solution spaces.

Analyzing Subspace Structures

- It provides insight into how subspaces intersect and combine, which is essential in the study of vector space decompositions and direct sums.

Example 1: Subspaces of $\mathbb{R}^3$

Suppose:

- U_1 is a plane in $\mathbb{R}^3$ with $\dim U_1 = 2$
- U_2 is a line in $\mathbb{R}^3$ with $\dim U_2 = 1$
- Their intersection $U_1 \cap U_2$ is either:

- The zero vector (if they are skew), with $\dim(U_1 \cap U_2) = 0$
- The line itself (if $U_2 \subseteq U_1$), with $\dim(U_1 \cap U_2) = 1$

Applying the formula:

- If $U_2 \subseteq U_1$:

$$
\dim(U_1 + U_2) = 2 + 1 - 1 = 2
$$

- If they intersect only at the zero vector:

$$
\dim(U_1 + U_2) = 2 + 1 - 0 = 3
$$

which shows how the intersection influences the dimension of the sum.

Conclusion: Significance of the Dimension Formula in Linear Algebra

The relationship between the dimensions of subspaces, their sum, and intersection encapsulated in the formula:

\[
\boxed{
\dim(U_1 + U_2) = \dim U_1 + \

Frequently Asked Questions

What is the formula relating the dimensions of subspaces U_1, U_2, their sum, and their intersection?

The formula is: $\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2)$.

Under what condition does $\dim(U_1 + U_2)$ equal $\dim(U_1) + \dim(U_2)$?

When U_1 and U_2 intersect only at the zero vector, i.e., $U_1 \cap U_2 = \{0\}$, then $\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2)$.

What does the notation U_1 U_2 represent in the context of subspaces?

U_1 U_2 typically denotes the sum of the subspaces, $U_1 + U_2$.

How can the dimension of the sum of two subspaces be computed if their intersection is known?

Using the formula: $\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2)$.

Why is the dimension formula for the sum of subspaces important in linear algebra?

It helps in understanding how subspaces combine and provides insights into the structure of vector spaces, especially in basis calculations and subspace decompositions.

Can the formula $\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2)$ be generalized to more than two subspaces?

Yes, for multiple subspaces, the formula involves inclusion-exclusion principles, extending the concept to sums and intersections of multiple subspaces.

What is the significance of the intersection $U_1 \cap U_2$ in

the dimension formula?

The intersection accounts for the common elements shared by U1 and U2, preventing double counting when summing their dimensions.

Is the dimension of the sum of subspaces always less than or equal to the sum of their individual dimensions?

Yes, because dim(U1 + U2) ≤ dim(U1) + dim(U2), with equality when U1 and U2 intersect only at the zero vector.

Additional Resources

Show That If U1 And U2 Are Subspaces Of A Vector Space V, Then
dim(U1 + U2) = Dim U1 + Dim U2 – Dim(U1 ∩ U2),
where

Understanding the Foundation: Subspaces and Their Significance

In the realm of linear algebra, the notion of subspaces plays a pivotal role. Subspaces are subsets of a vector space that themselves satisfy the axioms of a vector space. They are fundamental in breaking down complex vector spaces into manageable parts and are instrumental in understanding the structure of linear systems.

When dealing with multiple subspaces within a larger vector space \(V \), an essential question arises: how do the dimensions of these subspaces relate to the dimension of their sum? The formula that connects these quantities provides crucial insights into the structure and relationships of subspaces and is fundamental to many areas of mathematics and engineering.

This article delves into the theorem that relates the dimensions of two subspaces and their sum, including the intersection, guiding you through the proof, implications, and applications.

Preliminaries: Basic Concepts and Definitions

Before exploring the main theorem, it is important to understand some key concepts:

Subspace

A subset \(U \) of a vector space \(V \) over a field \(F \) is called a subspace if:
- The zero vector \(0 \) of \(V \) is in \(U \).

- U is closed under vector addition: if $u_1, u_2 \in U$, then $u_1 + u_2 \in U$.
- U is closed under scalar multiplication: if $u \in U$ and $\alpha \in F$, then $\alpha u \in U$.

Sum of Subspaces

Given two subspaces U_1 and U_2 of V, their sum $U_1 + U_2$ is defined as:
$$U_1 + U_2 = \{ u_1 + u_2 \mid u_1 \in U_1, u_2 \in U_2 \}$$
This sum is itself a subspace of V.

Intersection of Subspaces

The intersection $U_1 \cap U_2$ contains all vectors common to both U_1 and U_2. It is also a subspace of V.

Dimension

The dimension $\dim U$ of a subspace U is the number of vectors in its basis, i.e., the maximum set of linearly independent vectors in U.

The Core Theorem: Relating Dimensions

Theorem Statement:

If U_1 and U_2 are subspaces of a vector space V, then the dimension of their sum satisfies:

$$
\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)
$$

This formula captures the essence of how the subspaces overlap and how their union's size relates to their individual sizes.

Intuitive Explanation

Imagine two overlapping circles representing subspaces U_1 and U_2. The total area covered by both (their union) depends on the sizes of each circle minus the overlapping region to avoid double-counting. The theorem formalizes this intuition in the language of dimensions.

Proving the Dimension Formula

To understand the theorem thoroughly, let’s walk through a detailed proof.

Step 1: Establishing Bases

- Choose a basis $\mathcal{B}_1$ for U_1.
- Choose a basis $\mathcal{B}_2$ for U_2.

Since the intersection $U_1 \cap U_2$ is a subspace of both U_1 and U_2, it has a basis $\mathcal{B}_0$.

Key idea:
We will construct a basis for $U_1 + U_2$ by combining bases for U_1 and U_2, carefully avoiding redundancy.

Step 2: Extending the Basis of the Intersection

- Extend $\mathcal{B}_0$ to bases $\mathcal{B}_1'$ of U_1 and $\mathcal{B}_2'$ of U_2 such that:

$$
\mathcal{B}_1' = \mathcal{B}_0 \cup \mathcal{B}_1^{}
$$
$$
\mathcal{B}_2' = \mathcal{B}_0 \cup \mathcal{B}_2^{}
$$

where:
- $\mathcal{B}_1^{}$ is a basis for the remaining part of U_1 not in the intersection.
- $\mathcal{B}_2^{}$ is a basis for the remaining part of U_2 not in the intersection.

Observation:
Vectors in $\mathcal{B}_1^{}$ and $\mathcal{B}_2^{}$ are linearly independent of each other and of $\mathcal{B}_0$.

Step 3: Constructing a Basis for $U_1 + U_2$

- The set:

$$
\mathcal{B} = \mathcal{B}_1^{} \cup \mathcal{B}_2^{} \cup \mathcal{B}_0
$$

forms a basis for $U_1 + U_2$.

- The vectors in $\mathcal{B}_0$ are common to both subspaces; adding the vectors in $\mathcal{B}_1^{}$ and $\mathcal{B}_2^{}$ ensures the basis spans both subspaces without redundancy.

Result:
The dimension of $U_1 + U_2$ is:

$$
\dim(U_1 + U_2) = |\mathcal{B}_0| + |\mathcal{B}_1^{}| + |\mathcal{B}_2^{}|
$$

which simplifies to:

$$
\dim(U_1 + U_2) = \dim(U_1 \cap U_2) + (\dim U_1 - \dim(U_1 \cap U_2)) + (\dim U_2 - \dim(U_2 \cap U_2))
$$

or more directly:

$$
\boxed{
\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)
}
$$

Implications and Applications of the Theorem

Understanding this relationship between subspaces and their dimensions has vast implications across mathematics and applied sciences.

1. Subspace Decomposition

The formula allows mathematicians to analyze how overlapping subspaces combine, aiding in decomposing complex vector spaces into simpler, more manageable parts.

2. Linear Independence and Span

In systems of linear equations and data analysis, knowing the dimensions helps to determine the independence of vectors and the span of sets.

3. Signal Processing and Data Compression

In engineering, the theorem underpins techniques such as principal component analysis (PCA), where understanding the overlap of subspaces (e.g., signal and noise spaces) is crucial.

4. Coding Theory

The relationship guides the design of error-correcting codes by analyzing subspaces representing codewords.

5. Computational Efficiency

Algorithms that compute the sum and intersection of subspaces rely on these dimension relationships for performance optimization.

Special Cases and Further Insights

The theorem simplifies in specific contexts:

- If $U_1 \cap U_2 = \{0\}$, then the subspaces are disjoint save for the zero vector, and:

$$
\dim(U_1 + U_2) = \dim U_1 + \dim U_2
$$

- If $U_1 \subseteq U_2$, then:

$$
U_1 + U_2 = U_2
$$

and the formula reduces accordingly, confirming the consistency of the theorem.

Conclusion: The Power of Structural Understanding

The dimension formula for the sum of subspaces serves as a cornerstone in linear algebra, offering a clear and elegant way to quantify how subspaces overlap within a larger vector space. Its proof, rooted in basis construction and linear independence, exemplifies the beauty of mathematical reasoning and its capacity to reveal deep structural truths.

By mastering this theorem, students and practitioners gain a vital tool for dissecting complex vector spaces, enabling advances in theoretical insights and practical applications across science, engineering, and beyond. The interrelation of subspaces and their dimensions not only enriches our understanding of linear structures but also

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