Show That The Given Set Of Functions Is Orthogonal On The Indicated Interval. Find The Norm Of Each Function

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Orthogonality is a fundamental concept in mathematical analysis, particularly in the study of function spaces, Fourier series, and orthogonal polynomials. Demonstrating that a set of functions is orthogonal over a specified interval involves verifying that the inner product (integral of their product over that interval) equals zero for distinct functions. Additionally, calculating the norm of each function, which is derived from the inner product, provides insight into their behavior and normalization potential in various applications.

This comprehensive guide will walk you through the process of establishing the orthogonality of a given set of functions on a specific interval and computing their norms. By understanding the underlying principles and following structured steps, you can analyze many functions, including trigonometric, polynomial, and exponential functions.

Understanding Orthogonality of Functions

Before diving into the specifics, it’s vital to understand what orthogonality means in the context of functions.

Definition of Orthogonality

Two functions $f(x)$ and $g(x)$ are orthogonal over an interval $[a, b]$ with respect to a weight function $w(x)$ if their inner product is zero:

$$
\langle f, g \rangle = \int_a^b f(x) g(x) w(x) \, dx = 0,
$$

where:

- $w(x)$ is a positive weight function on $[a, b]$,
- The integral evaluates the "overlap" between the two functions.

If $w(x) = 1$, the functions are orthogonal in the standard sense; if $w(x)$ differs, they are orthogonal with respect to that weight.

Inner Product and Norm

- Inner product: $\langle f, g \rangle = \int_a^b f(x) g(x) w(x) \, dx$,
- Norm of a function: $\|f\| = \sqrt{\langle f, f \rangle}$.

The norm measures the "size" or "length" of a function in the function space, often used for normalization.

Steps to Show Orthogonality of a Set of Functions

The process involves systematic steps:

Step 1: Identify the Functions and Interval

- Clearly specify the functions $f_n(x)$ in the set.
- Determine the interval $[a, b]$ over which orthogonality is to be tested.
- Note the weight function $w(x)$, if any.

Step 2: Compute the Inner Product for Distinct Functions

- Calculate $\langle f_m, f_n \rangle = \int_a^b f_m(x) f_n(x) w(x) \, dx$ for $m \neq n$.
- Use techniques such as substitution, integration by parts, or known integrals, depending on the functions.

Step 3: Confirm Orthogonality

- If $\langle f_m, f_n \rangle = 0$ for all $m \neq n$, then the set is orthogonal over $[a, b]$.

Step 4: Compute the Norm of Each Function

- Calculate $\|f_n\| = \sqrt{\langle f_n, f_n \rangle}$.
- These norms are essential for normalization or further analysis.

Common Examples of Orthogonal Function Sets

Many classical functions form orthogonal sets under specific intervals and weight functions. Understanding these examples helps in practice.

Trigonometric Functions: Fourier Series

- Over $[-\pi, \pi]$, the functions $\{\sin nx, \cos nx\}$ are orthogonal with respect to $w(x) = 1$.
- Inner product: $\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0$ for $n \neq m$.

Legendre Polynomials

- Defined on $[-1, 1]$.
- Orthogonal with weight $w(x) = 1$.
- Inner product: $\int_{-1}^1 P_m(x) P_n(x) \, dx = 0$ for $m \neq n$.

Chebyshev and Hermite Polynomials

- Each set has specific orthogonality intervals and weights, useful in approximation theory and quantum mechanics.

Worked Example: Orthogonality of Sine Functions on $[- \pi, \pi]$

Suppose we are given the set:

$$
\{f_n(x) = \sin nx \mid n = 1, 2, 3, \dots\}
$$

and want to show they are orthogonal over $[- \pi, \pi]$.

Step 1: Define the inner product

$$
\langle f_m, f_n \rangle = \int_{-\pi}^\pi \sin mx \sin nx \, dx
$$

Step 2: Compute the inner product for $m \neq n$

Use the product-to-sum identity:

$$
\sin mx \sin nx = \frac{1}{2} [\cos (m - n) x - \cos (m + n) x]
$$

Thus,

$$
\langle f_m, f_n \rangle = \frac{1}{2} \int_{-\pi}^\pi \left[\cos (m - n) x - \cos (m + n) x\right] dx
$$

Integrate term-by-term:

$$
\int_{-\pi}^\pi \cos k x \, dx =
\begin{cases}
2\pi, & \text{if } k=0 \\
0, & \text{if } k \neq 0
\end{cases}
$$

- When $m \neq n$, both $m - n \neq 0$ and $m + n \neq 0$, so both integrals are zero.

Therefore,

$$
\langle f_m, f_n \rangle = 0, \quad m \neq n
$$

Conclusion: The sine functions are orthogonal over $[- \pi, \pi]$.

Step 3: Compute the norm of each function

$$
\|f_n\|^2 = \langle f_n, f_n \rangle = \int_{-\pi}^\pi \sin^2 nx \, dx
$$

Using the identity $\sin^2 nx = \frac{1 - \cos 2 nx}{2}$:

$$
\|f_n\|^2 = \frac{1}{2} \int_{-\pi}^\pi [1 - \cos 2 nx] dx = \frac{1}{2} \left[\int_{-\pi}^\pi 1 \, dx - \int_{-\pi}^\pi \cos 2 nx \, dx \right]
$$

Calculate each:

$$
\int_{-\pi}^\pi 1 \, dx = 2\pi
$$

$$
\int_{-\pi}^\pi \cos 2 nx \, dx = 0 \quad (\text{for } n \neq 0)
$$

Thus,

$$
\|f_n\|^2 = \frac{1}{2} (2\pi) = \pi
$$

The norm:

$$
\boxed{
\|f_n\| = \sqrt{\pi}
}
$$

This shows each sine function has the same norm over $[- \pi, \pi]$.

Applying to Other Sets of Functions

The process outlined above extends to various other function sets, often with modifications according to the specific properties of the functions and their intervals.

1. Polynomial Sets (Legendre, Chebyshev, etc.)

- Use their specific orthogonality relations.
- Integrate with the corresponding weight function over the defined interval.

2. Exponential Functions

- Typically orthogonal over semi-infinite intervals with exponential weights.
- Application in Laplace transforms and signal processing.

3. Bessel and Other Special Functions

- Frequently used in solving differential equations.
- Orthogonality properties depend on boundary conditions and weight functions.

Practical Tips for Proving Orthogonality and Computing Norms

- Use known identities: Leveraging product-to-sum formulas and integral identities simplifies calculations.
- Check boundary conditions: Confirm that functions satisfy conditions needed for certain orthogonality properties.
- Leverage symmetry: Symmetric integrals often evaluate to zero, aiding in orthogonality proofs.
- Use tables of integrals: Standard integral results can save time.
- Normalize functions: Once norms are known, normalize functions as needed for orthonormal bases.

Summary and Key Takeaways

- To demonstrate that a set of functions is orthogonal over an interval, compute the

Frequently Asked Questions

How can we show that the set of functions $\{\sin nx\}_{n=1}^\infty$ is orthogonal on the interval $[0, \pi]$?

To show orthogonality, compute the integral $\int_0^{\pi} \sin nx \sin mx \; dx$. When $n \neq m$, this integral equals zero, confirming orthogonality. For $n = m$, the integral yields $\frac{\pi}{2}$, which helps in finding the norm.

Given the set of functions $\{1, \cos x, \cos 2x, \cos 3x \}$ on $[0, 2\pi]$, how do we verify their orthogonality?

Calculate the integrals $\int_0^{2\pi} \cos nx \cos mx \; dx$. The integral is zero when $n \neq m$ and equals π when $n = m \neq 0$, thus confirming orthogonality among these functions.

What is the process to find the norm of $\sin nx$ on $[0, \pi]$?

The norm is given by $\| \sin nx \| = \sqrt{ \int_0^{\pi} \sin^2 nx \; dx }$. Evaluating this integral yields $\sqrt{ \frac{\pi}{2} }$, which is the norm of the function.

How do we determine if a set of functions $\{ \cos nx \}$ is orthogonal over $[-\pi, \pi]$?

Compute $\int_{-\pi}^{\pi} \cos nx \cos mx \; dx$. For $n \neq m$, the integral is zero, confirming orthogonality. When $n = m$, the integral is π, which can be used to find the norm.

Can you explain how to find the norm of $\cos nx$ on $[0, \pi]$?

Yes, evaluate $\sqrt{ \int_0^{\pi} \cos^2 nx \; dx }$. The integral evaluates to $\frac{\pi}{2}$, so the norm is $\sqrt{ \frac{\pi}{2} }$.

What are the key steps to verify the orthogonality of a given set of functions in Fourier analysis?

The main steps are: (1) set up the integral of the product of two distinct functions over the interval, (2) evaluate the integral; if it equals zero for distinct functions, they are orthogonal, and (3) compute the integral of the square of each function to find their norms.

How does knowing the norms of the functions help in orthogonal set analysis?

Knowing the norms allows us to normalize the functions, turning the set into an orthonormal set, which simplifies Fourier coefficient calculations and expansion of functions in terms of these basis functions.

For the functions $\{ \sin nx, \cos nx \}$ on $[0, 2\pi]$, how do we confirm they form an orthogonal set?

Calculate integrals such as $\int_0^{2\pi} \sin nx \cos mx \; dx$ and $\int_0^{2\pi} \sin nx \sin mx \; dx$. The first is zero for all n, m, and the second is zero when $n \neq m$. When $n = m$, the integral yields the norm squared, confirming orthogonality.

Additional Resources

Orthogonality of Function Sets and Computation of Norms: A Comprehensive Analysis

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Introduction

In the realm of functional analysis and mathematical physics, the concept of orthogonality among functions plays a pivotal role. It extends the familiar notion of perpendicular vectors in Euclidean space to infinite-dimensional function spaces. Recognizing orthogonal sets of functions allows us to simplify complex problems, especially in solving differential equations, expanding functions into series, and analyzing signals. This review aims to delve into the process of establishing that a given set of functions is orthogonal over a specific interval, alongside calculating the norms of each function within that set.

Fundamental Concepts: Orthogonality and Norms

What Is Orthogonality in Function Spaces?

In the context of function spaces, two functions $f(x)$ and $g(x)$ are said to be orthogonal over an interval $[a, b]$ with respect to a weight function $w(x)$ if their inner product is zero:

$$
\boxed{
\langle f, g \rangle = \int_{a}^{b} f(x) g(x) w(x) \, dx = 0
}
$$

- Inner Product: The integral defining the inner product generalizes the dot product between vectors.
- Weight Function: $w(x)$ adjusts the importance of different parts of the interval; in many classical cases, $w(x) = 1$.

Norm of a Function

The norm of a function $f(x)$, denoted $\|f\|$, quantifies its 'size' or 'magnitude' in the function space:

$$
\boxed{
\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_{a}^{b} [f(x)]^2 w(x) \, dx}
}
$$

- The norm plays a crucial role in normalization, orthogonal expansions, and convergence analysis.

Establishing Orthogonality: Step-by-Step Approach

Step 1: Identify the Set of Functions and Interval

- Given Functions: Specify the functions $f_1(x), f_2(x), \ldots, f_n(x)$.
- Interval: Clearly define the domain $[a, b]$ over which the functions are considered.
- Weight Function: Note if a weight function $w(x)$ is involved; for many classical orthogonal function systems, $w(x) = 1$.

Step 2: Write Down the Inner Product

For each pair (f_i, f_j), compute:

\[

$$
\langle f_i, f_j \rangle = \int_{a}^{b} f_i(x) f_j(x) w(x) \, dx
$$

- When $i \neq j$, if the integral evaluates to zero, functions are orthogonal.
- When $i = j$, the integral gives the square of the norm of f_i.

Step 3: Perform the Integrations

- Use techniques like substitution, partial fractions, integration by parts, or known integral formulas.
- Leverage properties of special functions (e.g., Legendre polynomials, Chebyshev polynomials, sines, cosines) which often have known orthogonality relations.

Step 4: Confirm Orthogonality

- For all pairs (f_i, f_j) with $i \neq j$, verify that $\langle f_i, f_j \rangle = 0$.
- If this holds, the set is orthogonal over $[a, b]$.

Computing Norms of Functions

Once orthogonality is established, calculating the norm of each function involves:

$$
\|f_i\| = \sqrt{\int_{a}^{b} [f_i(x)]^2 w(x) \, dx}
$$

- This integral provides the 'length' of the function in the function space.
- Norms are used to normalize functions, transforming them into orthonormal sets when divided by their norms.

Classical Examples and Their Significance

1. Fourier Series: Sines and Cosines

- Interval: $[- \pi, \pi]$
- Weight: $w(x) = 1$
- Functions: $\sin(nx)$, $\cos(nx)$

Orthogonality Relations:

$$
\int_{-\pi}^{\pi} \cos(mx) \cos(nx) \, dx =
\begin{cases}
2\pi, & m = n \neq 0 \\
\pi, & m = n = 0 \\
0, & m \neq n
\end{cases}
$$

$$
\int_{-\pi}^{\pi} \sin(mx) \sin(nx) \, dx =
\begin{cases}
2\pi, & m = n \neq 0 \\
0, & m \neq n
\end{cases}
$$

Norms:

$$
\|\cos(nx)\| = \sqrt{\int_{-\pi}^{\pi} \cos^2(nx) \, dx}
= \sqrt{\pi}
$$

$$
\|\sin(nx)\| = \sqrt{\int_{-\pi}^{\pi} \sin^2(nx) \, dx}
= \sqrt{\pi}
$$

These results underpin Fourier analysis, enabling expansion of arbitrary periodic functions into sines and cosines.

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2. Legendre Polynomials

- Interval: $[-1, 1]$
- Weight: $w(x) = 1$

Orthogonality:

$$
\int_{-1}^1 P_m(x) P_n(x) \, dx = 0, \quad m \neq n
$$

Norms:

$$
\|P_n\|^2 = \frac{2}{2n + 1}
$$

This orthogonality is fundamental in solving Laplace's equation in spherical coordinates and in expanding functions in Legendre series.

Advanced Considerations

Generalizations and Applications

- Weighted Orthogonality: Many function systems, such as Jacobi, Laguerre, and Hermite polynomials, are orthogonal with respect to specific weight functions over infinite or finite intervals.
- Eigenfunction Expansions: Orthogonal functions often arise as eigenfunctions of Sturm-Liouville problems, ensuring orthogonality properties.
- Orthogonalization Procedures: When a set of functions isn't orthogonal, processes like Gram-Schmidt can construct an orthogonal basis.

Numerical Integration and Approximation

- Accurate computation of integrals is essential, especially when analytical solutions are infeasible.
- Numerical methods, such as Gaussian quadrature, leverage orthogonal polynomials for efficient integration.

Practical Examples and Step-by-Step Calculations

Example 1: Show that $\sin(nx)$ and $\sin(mx)$ are orthogonal on $[0, \pi]$

- Interval: $[0, \pi]$
- Weight: $w(x) = 1$

Compute:

$$
\langle \sin(nx), \sin(mx) \rangle = \int_0^\pi \sin(nx) \sin(mx) \, dx
$$

Using trigonometric identities:

$$
\sin(nx)\sin(mx) = \frac{1}{2}[\cos(n - m)x - \cos(n + m)x]
$$

Integrate term-by-term:

$$
\int_0^\pi \cos(kx) \, dx =
\begin{cases}
0, & k \neq 0 \\
\pi, & k=0
\end{cases}
$$

- For $n \neq m$:

$$
\langle \sin(nx), \sin(mx) \rangle = 0
$$

- For $n = m \neq 0$:

$$
\int_0^\pi \sin^2(nx) \, dx = \frac{\pi}{2}
$$

Thus, these functions are orthogonal, and their norms are $\sqrt{\pi/2}$.

Example 2: Orthogonality of Legendre Polynomials

To verify $P_2(x)$ and $P_3(x)$ are orthogonal:

- Functions:

$$
P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x)
$$

- Interval: $[-1, 1]$

Compute:

$$
\int_{-1}^1 P_2(x) P_3(x) \, dx
$$

- Expand and multiply:

$$
P_2(x) P_3(x) = \frac{1}{4}(3x^2 - 1)(5x^3 - 3x)
$$

- Expand numerator:

$$
= \frac{1}{4}[15x^5 - 9x^3 - 5x^3 + 3x]
= \frac{1}{4}(15x^5 - 14x^3 + 3x)
$$

- Integrate term-by-term over $[-1, 1]$:

$$
\int_{-1}^1 x
$$

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