Solve The Differential Equation Using Laplace Transforms. The Solution Is Y(t) And Y(t) Y" 2y 8y = 3t+26(t),

Solve The Differential Equation Using Laplace Transforms. The Solution Is Y(t) And Y(t) Y" 2y 8y = 3t+26(t),

Solving differential equations is a fundamental skill in engineering, physics, and applied mathematics. Among the various methods available, Laplace transforms stand out for their efficiency in handling linear differential equations, especially those with initial conditions and non-homogeneous terms. In this article, we will explore how to solve the differential equation:

$$ Y''(t) - 2Y(t) + 8Y(t) = 3t + 26(t) $$

using Laplace transforms. Our goal is to find the solution $Y(t)$, which describes the behavior of the system described by the differential equation.

Understanding the Differential Equation

Before diving into the Laplace transform technique, it's crucial to understand the structure of the differential equation at hand.

Equation Breakdown

- The differential equation involves the second derivative $Y''(t)$, indicating a second-order linear differential equation.
- The terms involve $Y(t)$ and its derivatives, with coefficients that are constants.
- The right-hand side (forcing function) is a combination of a polynomial $3t$ and an exponential function $26(t)$.

Standard Form

Rearranged, the differential equation can be written as:

$$ Y''(t) - 2Y(t) + 8Y(t) = 3t + 26(t) $$

which simplifies to:

$$ Y''(t) + 6Y(t) = 3t + 26(t) $$

Applying Laplace Transforms

Laplace transforms are powerful tools for converting differential equations from the time domain into the algebraic domain, simplifying the process of solving for $Y(t)$.

Step 1: Take the Laplace Transform of Both Sides

Recall the Laplace transform of derivatives:

- $\mathcal{L}\{Y'(t)\} = sY(s) - Y(0)$
- $\mathcal{L}\{Y''(t)\} = s^2Y(s) - sY(0) - Y'(0)$

Assuming initial conditions $Y(0) = Y_0$ and $Y'(0) = Y_1$, which are typically known or assumed zero if not specified.

Applying the Laplace transform to the differential equation:

$$ \mathcal{L}\{Y''(t)\} + 6\mathcal{L}\{Y(t)\} = \mathcal{L}\{3t + 26(t)\} $$

becomes:

$$ s^2Y(s) - sY_0 - Y_1 + 6Y(s) = \frac{3}{s^2} + \frac{26}{s} $$

Step 2: Solve for $Y(s)$

Rearranged, the algebraic equation in the s-domain is:

$$ (s^2 + 6)Y(s) = sY_0 + Y_1 + \frac{3}{s^2} + \frac{26}{s} $$

Thus,

$$ Y(s) = \frac{sY_0 + Y_1}{s^2 + 6} + \frac{3}{s^2(s^2 + 6)} + \frac{26}{s(s^2 + 6)} $$

This expression contains terms that can be further simplified using partial fraction decomposition to facilitate inverse Laplace transforms.

Partial Fraction Decomposition

To invert $Y(s)$ back to $Y(t)$, we need to decompose the complex fractions into simpler parts.

Breaking Down the Terms

1. Term 1: $\frac{sY_0 + Y_1}{s^2 + 6}$

2. Term 2: $\frac{3}{s^2(s^2 + 6)}$

3. Term 3: $\frac{26}{s(s^2 + 6)}$

Partial Fractions for Term 2

Express:

$$ \frac{3}{s^2(s^2 + 6)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 6} $$

Multiply both sides by $s^2(s^2 + 6)$:

$$ 3 = A s(s^2 + 6) + B (s^2 + 6) + (C s + D) s^2 $$

Solve for constants A, B, C, D by plugging in convenient values of s or equating coefficients.

Similarly, decompose Term 3:

$$ \frac{26}{s(s^2 + 6)} = \frac{E}{s} + \frac{F s + G}{s^2 + 6} $$

Inverse Laplace Transform to Find $Y(t)$

Once the partial fractions are determined, apply known inverse Laplace transforms:

- $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$
- $\mathcal{L}^{-1}\left\{\frac{1}{s^2 + a^2}\right\} = \frac{\sin a t}{a}$
- $\mathcal{L}^{-1}\left\{\frac{s}{s^2 + a^2}\right\} = \cos a t$
- $\mathcal{L}^{-1}\left\{\frac{1}{s^2 + a^2}\right\} = \frac{\sin a t}{a}$

These transforms will produce functions involving sines, cosines, and polynomials, leading to the explicit form of $Y(t)$.

Complete Solution $Y(t)$

Putting all pieces together, the general solution $Y(t)$ will be composed of:

- The homogeneous solution $Y_h(t)$ from the complementary equation:

$$Y''(t) + 6Y(t) = 0$$

which has the solution:

$$Y_h(t) = C_1 \cos (\sqrt{6} t) + C_2 \sin (\sqrt{6} t)$$

- The particular solution $Y_p(t)$ obtained from the inverse Laplace transform of the non-homogeneous part.

The full solution:

$$Y(t) = Y_h(t) + Y_p(t)$$

with constants C_1 and C_2 determined by initial conditions.

Summary of the Solution Process

To recap, solving the differential equation using Laplace transforms involves these key steps:

1. Write the differential equation in standard form.
2. Take the Laplace transform of both sides, incorporating initial conditions.
3. Solve the resulting algebraic equation for $Y(s)$.
4. Use partial fraction decomposition to simplify complex fractions.
5. Find the inverse Laplace transform of each term.
6. Combine the inverse transforms to obtain $Y(t)$.
7. Apply initial conditions to solve for constants in the homogeneous solution.

Conclusion

Using Laplace transforms offers a systematic approach to solving linear differential equations, especially when dealing with non-homogeneous terms like polynomials and exponentials. By transforming the differential equation into an algebraic form, applying partial fractions, and then performing inverse transforms, we can obtain explicit solutions for $Y(t)$.

This method is particularly useful in engineering applications such as control systems, circuit analysis, and mechanical vibrations, where initial conditions are crucial and solutions often involve complex forcing functions.

Additional Tips for Solving Differential Equations Using Laplace Transforms

- Always verify initial conditions before solving.
- Practice partial fraction decomposition to handle complex rational functions.
- Use standard Laplace transform tables for inverse transforms.
- Check your solution by differentiating $Y(t)$ and substituting back into the original differential equation.

By mastering these steps, you can efficiently solve a wide range of differential equations and analyze the dynamic behavior of various systems.

Meta Description:
Learn how to solve the differential equation \(Y"(t) - 2Y(t) + 8Y(t) = 3t + 26(t) \) using Laplace transforms. This comprehensive guide covers the step-by-step process, partial fractions, inverse transforms, and solutions, ideal for students and engineers.

Frequently Asked Questions

What is the main advantage of using Laplace transforms to solve differential equations like Y" - 2Y + 8Y = 3t + 26t?

Laplace transforms convert differential equations into algebraic equations, making them easier to solve by handling initial conditions and complex derivatives systematically.

How do you apply Laplace transforms to the differential equation Y" - 2Y' + 8Y = 3t + 26t?

Apply the Laplace transform to each term, using linearity and known transforms for derivatives: L{Y"} = s^2Y(s) - sY(0) - Y'(0), then rearrange to solve for Y(s).

What initial conditions are needed to solve the differential equation using Laplace transforms?

Initial conditions such as Y(0) and Y'(0) are required to fully determine the solution after transforming the differential equation.

How do you handle the right-hand side functions 3t and 26t when applying Laplace transforms?

Use known Laplace transforms: L{t} = 1/s^2, so L{3t} = 3/s^2 and L{26t} = 26/s^2, then include these in the algebraic equation.

What is the general form of the solution after taking the inverse Laplace transform in this problem?

The solution Y(t) will be expressed as a combination of exponential, polynomial, and possibly sinusoidal functions depending on the inverse transform applied to Y(s).

Can you explain the step-by-step process to find Y(t) for the differential equation given?

Yes, the steps are: 1) Take Laplace transform of both sides, 2) Solve for Y(s), 3) Simplify the algebraic expression, 4) Use inverse Laplace transform to find Y(t).

What common challenges might arise when solving this differential equation using Laplace transforms?

Challenges include correctly applying initial conditions, performing partial fraction decomposition, and inverting complex rational expressions accurately.

How does the linearity property of Laplace transforms simplify solving differential equations?

Linearity allows you to transform each term separately and then combine solutions, simplifying the handling of multiple terms on the right-hand side.

What are the typical forms of the solutions Y(t) for differential equations like Y'' - 2Y' + 8Y = 3t + 26t?

The solutions often involve exponential functions, polynomial terms, and possibly sinusoidal functions, reflecting the roots of the characteristic equation and particular solutions.

Additional Resources

Solve The Differential Equation Using Laplace Transforms. The Solution Is Y(t) And Y(t) Y" 2y 8y = 3t+26(t)

Introduction

Differential equations are fundamental tools in mathematics and engineering, modeling phenomena ranging from mechanical vibrations to electrical circuits. Among various methods to solve these equations, the Laplace Transform stands out for its efficiency, especially when dealing with initial value problems involving complex derivatives and forcing functions. In this article, we explore the process of solving a second-order linear differential equation using Laplace transforms. The specific equation in focus is:

\[Y''(t) - 2Y(t) + 8Y(t) = 3t + 26(t) \]

This form combines derivatives with algebraic functions, presenting a perfect candidate to demonstrate the power of Laplace transforms in simplifying and solving such problems.

Understanding the Differential Equation

Before diving into the solution process, let's clarify the differential equation:

$$ Y''(t) - 2Y(t) + 8Y(t) = 3t + 26(t) $$

At first glance, the equation appears to have some inconsistencies in notation, notably the term "26(t)." Typically, in differential equations, the right side involves functions of t, such as polynomials or exponential functions. Assuming the notation "26(t)" is a typographical error or shorthand for a specific function, perhaps the problem intends to include a function like $26 \sin t$ or similar.

For clarity, we will interpret the right-hand side as:

$$ 3t + 26 \sin t $$

This assumption aligns with standard differential equations involving polynomial and sinusoidal forcing functions. If the problem intends something else, the methodology remains similar, but the functions involved may differ.

So, our corrected differential equation becomes:

$$ Y''(t) + 6Y(t) = 3t + 26 \sin t $$

where we've combined like terms ($-2Y(t) + 8Y(t) = 6Y(t)$).

Step 1: Applying the Laplace Transform

The first step is to take the Laplace transform of both sides of the differential equation. Recall that the Laplace transform of derivatives is:

- $\mathcal{L}\{Y'(t)\} = sY(s) - Y(0)$
- $\mathcal{L}\{Y''(t)\} = s^2Y(s) - sY(0) - Y'(0)$

Assuming initial conditions are zero for simplicity (i.e., $Y(0) = 0$ and $Y'(0) = 0$), we have:

$$ \mathcal{L}\{Y''(t)\} = s^2Y(s) $$

Applying the Laplace transform to both sides:

$$ s^2Y(s) + 6Y(s) = \mathcal{L}\{3t + 26 \sin t\} $$

Step 2: Transforming the Right-Hand Side

Next, we compute the Laplace transforms of each term on the right:

- $\mathcal{L}\{3t\} = 3 \times \frac{1}{s^2}$

- $\mathcal{L}\{26 \sin t\} = 26 \times \frac{1}{s^2 + 1}$

Therefore, the right side becomes:

$$ \frac{3}{s^2} + \frac{26}{s^2 + 1} $$

Step 3: Solving for $Y(s)$

Rearranging the transformed equation:

$$ (s^2 + 6)Y(s) = \frac{3}{s^2} + \frac{26}{s^2 + 1} $$

Hence,

$$ Y(s) = \frac{\frac{3}{s^2} + \frac{26}{s^2 + 1}}{s^2 + 6} $$

Expressed as a single fraction:

$$ Y(s) = \frac{3}{s^2 (s^2 + 6)} + \frac{26}{(s^2 + 1)(s^2 + 6)} $$

Now, the task is to find the inverse Laplace transform of $Y(s)$. To do so, we'll decompose $Y(s)$ into partial fractions.

Step 4: Partial Fraction Decomposition

Let's handle each term separately.

Term 1:

$$ \frac{3}{s^2 (s^2 + 6)} $$

Set:

$$ \frac{3}{s^2 (s^2 + 6)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C s + D}{s^2 + 6} $$

Multiply both sides by the denominator:

$$ 3 = A s (s^2 + 6) + B (s^2 + 6) + (C s + D) s^2 $$

Expanding:

$$ 3 = A s^3 + 6A s + B s^2 + 6 B + C s^3 + D s^2 $$

Group like terms:

$$ 3 = (A + C) s^3 + (B + D) s^2 + 6A s + 6 B $$

Equate coefficients on both sides:

- For $ s^3 $: $ A + C = 0 $

- For $ s^2 $: $ B + D = 0 $

- For $ s^1 $: $ 6A = 0 \Rightarrow A = 0 $

- For constant term: $ 6 B = 3 \Rightarrow B = \frac{1}{2} $

From $ A = 0 $, $ A + C = 0 \Rightarrow C = 0 $.

From $ B = \frac{1}{2} $, $ B + D = 0 \Rightarrow D = -\frac{1}{2} $.

So, the partial fractions are:

$$ \frac{3}{s^2 (s^2 + 6)} = \frac{0}{s} + \frac{1/2}{s^2} + \frac{0 \cdot s - 1/2}{s^2 + 6} $$

Simplify:

$$ \frac{3}{s^2 (s^2 + 6)} = \frac{1/2}{s^2} - \frac{1/2}{s^2 + 6} $$

Term 2:

$$ \frac{26}{(s^2 + 1)(s^2 + 6)} $$

Use partial fractions:

$$ \frac{26}{(s^2 + 1)(s^2 + 6)} = \frac{E}{s^2 + 1} + \frac{F}{s^2 + 6} $$

Multiply both sides:

$$ 26 = E (s^2 + 6) + F (s^2 + 1) $$

$$ 26 = (E + F) s^2 + 6E + F $$

Matching coefficients:

- s^2: $E + F = 0 \Rightarrow F = -E$

- Constant term: $6E + F = 26$

Substitute $F = -E$:

$$ 6E - E = 26 \Rightarrow 5E = 26 \Rightarrow E = \frac{26}{5} $$

Then,

$$ F = -\frac{26}{5} $$

So,

$$ \frac{26}{(s^2 + 1)(s^2 + 6)} = \frac{26/5}{s^2 + 1} - \frac{26/5}{s^2 + 6} $$

Step 5: Constructing $Y(s)$

Putting it all together:

$$
Y(s) = \left(\frac{1/2}{s^2} - \frac{1/2}{s^2 + 6} \right) + \left(\frac{26/5}{s^2 + 1} - \frac{26/5}{s^2 + 6} \right)
$$

Group similar terms:

\[
Y(s) = \frac{1/2}{s^2} + \frac{26/5}{s^2 + 1} - \left(\frac{1/2}{s^2 + 6} + \frac{26/5}{s^2 + 6} \right)
\]

Simplify the last parentheses:

\[
\frac{1/2 + 26/5}{s^2 + 6}
\]

Calculate numerator:

\[\frac{1/2 + 26/5}{s^2 + 6} = \frac{\frac{5}{10} + \frac{52}{10}}{s^2 + 6} = \frac{\frac{57}{10}}{s^2 + 6} \]

Therefore, the final form:

\[
Y(s) = \frac{1/2}{s^2} + \

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