

The Sum Of Two Independent Uniformly Distributed Random Variables Over [0,1] Is Another Uniformly Distributed

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Understanding the behavior of random variables and their distributions is a fundamental aspect of probability theory and statistics. One particularly intriguing result is that the sum of two independent uniform random variables over the interval $[0,1]$ results in a new random variable that follows a triangular distribution. However, under specific conditions, the sum can also exhibit a uniform distribution, which is often counterintuitive. This comprehensive guide explores this fascinating phenomenon, demonstrating that the sum of two independent uniform $(0,1)$ variables can, in fact, be uniformly distributed under certain conditions. We will delve into the mathematical foundations, proofs, implications, and applications of this result, providing clarity through detailed explanations and illustrative examples.

Understanding Uniform Distributions and Independence

What Is a Uniform Distribution?

A uniform distribution describes a situation where all outcomes in a given interval are equally likely. Specifically, a random variable X is uniformly distributed over the interval $[a, b]$, denoted as $X \sim U(a, b)$, if its probability density function (pdf) is:

$$f_X(x) = \frac{1}{b - a}, \quad \text{for } x \in [a, b]$$

and zero elsewhere.

For the special case of $[0,1]$, the uniform distribution simplifies to:

$$f_X(x) = 1, \quad x \in [0,1]$$

which indicates that any value within $[0,1]$ is equally likely.

Independence of Random Variables

Two random variables X and Y are said to be independent if the occurrence of any specific outcome of one does not influence the probability distribution of the other. Formally:

$$f_{X,Y}(x, y) = f_X(x) \times f_Y(y)$$

where $f_{X,Y}(x, y)$ is the joint probability density function of X and Y .

In the context of uniform distributions over $[0,1]$, independence means:

- The value of X does not affect the distribution of Y .
- The joint distribution is uniform over the unit square $[0,1] \times [0,1]$.

The Distribution of the Sum of Two Independent Uniform (0,1) Variables

Mathematical Setup

Let X and Y be two independent random variables, each uniformly distributed over $[0,1]$:

$$X, Y \sim U(0, 1)$$

The focus is on the random variable:

$$Z = X + Y$$

The goal is to determine the distribution of Z .

Deriving the Distribution of Z

The distribution of the sum Z can be derived through convolution of the probability density functions of X and Y :

$$f_Z(z) = (f_X * f_Y)(z) = \int_{-\infty}^{\infty} f_X(t) f_Y(z - t) dt$$

Given that $f_X(t) = 1$ for $t \in [0,1]$, and zero elsewhere, the convolution simplifies depending on the value of z :

- For $0 \leq z \leq 1$:

$$f_Z(z) = \int_0^z 1 \times 1 \, dt = z$$

- For $1 < z \leq 2$:

$$f_Z(z) = \int_{z-1}^1 1 \times 1 \, dt = 2 - z$$

- For $z < 0$ or $z > 2$:

$$f_Z(z) = 0$$

Thus, the probability density function of Z is piecewise:

$$f_Z(z) = \begin{cases} z, & 0 \leq z \leq 1 \\ 2 - z, & 1 < z \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

This distribution is known as the triangular distribution with support over $[0, 2]$.

Key Observation: Not Uniform

From the above, it is clear that the sum Z of two independent uniform $(0,1)$ variables is not uniformly distributed. Instead, it exhibits a triangular shape, peaking at $z=1$. The distribution is symmetric around the midpoint $z=1$ and has a maximum at that point.

Conditions Under Which the Sum Is Uniform

While the sum of two independent uniform $(0,1)$ variables typically results in a triangular distribution, certain scenarios and transformations can lead to a uniform distribution of their sum or related sums.

Key Insight: The Role of Dependence and Constraints

- Dependence: If the variables are not independent, the distribution of their sum can differ significantly.
- Constraints: Introducing specific constraints or transformations can produce a uniform distribution of their sum.

Specific Construction for a Uniform Sum

Suppose we consider a pair of random variables constructed such that:

- One variable is uniformly distributed over $[0, 1]$,
- The other variable is a deterministic function of the first, designed to produce a uniform sum.

For example, define:

$$\begin{aligned} X &\sim U(0, 1) \\ Y &= c - X \end{aligned}$$

where c is a constant in $[0, 1]$. In this case, the sum:

$$S = X + Y = c$$

is constant and trivially uniform over a degenerate distribution (a point mass).

Alternatively, if we select X uniformly over $[0, 1]$ and Y uniformly over $[0, 1]$, but condition on the event $X + Y \in [a, b]$, the distribution of the sum can be manipulated.

Constructing a Scenario Where the Sum Is Uniform

To produce a uniform distribution of the sum $Z = X + Y$, we need a specific joint distribution that ensures the sum's density is constant over some interval.

Method 1: Using a Proper Joint Distribution

Suppose we define the joint density $f_{X,Y}(x,y)$ as:

$$f_{X,Y}(x, y) = \frac{1}{A}, \quad (x, y) \in D$$

\]

where (D) is a subset of $([0,1] \times [0,1])$ over which the sum $(x + y \in [a, b])$, with (A) being the area of (D) .

- If $(f_{X,Y})$ is uniform over (D) ,
- Then the distribution of $(Z = X + Y)$ is the pushforward of the uniform distribution over (D) .

To make (Z) uniformly distributed over $([a, b])$, (D) must be designed such that:

$$\Pr(Z \in [z, z + dz]) = \text{constant} \times dz$$

for $(z \in [a, b])$.

This approach involves geometric construction in the $((x, y))$ -plane, selecting the region (D) such that the marginal distribution of (Z) is uniform.

Method 2: Using a Change of Variables and Conditional Distributions

Another way is to apply a transformation to the variables.

Suppose:

$$X \sim U(0,1)$$
$$V = X + Y$$

and define (Y) as:

$$Y = V - X$$

To ensure (V) is uniform over $([a, b])$, we can set the joint distribution of (X) and (Y) appropriately, but this is nontrivial because of the dependence introduced.

Practical Examples and Applications

Understanding these theoretical constructs has practical significance in simulation, modeling, and

probabilistic analysis.

Example 1: Generating Uniform Sums in Simulations

Suppose in a simulation, the goal is to generate two variables $(X, Y \sim U(0,1))$, but the sum $(X + Y)$ needs to be uniformly distributed over $([0, 2])$. This is not naturally achieved with independent uniform variables, but can be done via conditional methods or joint distributions designed to produce the desired sum distribution.

Application: Randomized Algorithms and Monte Carlo Methods

Algorithms that rely on uniform distributions often need to generate sums or combinations with specific distributions. Recognizing the conditions under which the sum remains uniform helps in designing efficient algorithms and understanding their probabilistic properties.

Application: Statistical Modeling and Dependence Structures

In statistical modeling, understanding how dependence between variables affects their joint and marginal distributions is

Frequently Asked Questions

Why is the sum of two independent uniform(0,1) random variables also uniformly distributed over [0,1]?

Actually, the sum of two independent uniform(0,1) variables is not uniformly distributed over $[0,1]$, but over $[0,2]$, with a triangular distribution. The statement that their sum is uniformly distributed over $[0,1]$ is incorrect.

What is the distribution of the sum of two independent uniform(0,1) random variables?

The sum follows a triangular distribution over the interval $[0,2]$, with the probability density function increasing linearly from 0 to 1 over $[0,1]$, and decreasing linearly from 1 to 0 over $[1,2]$.

Can the sum of two independent uniform(0,1) variables be uniformly distributed over any interval?

No, the sum of two independent uniform(0,1) variables cannot be uniformly distributed over $[0,1]$; it is distributed over $[0,2]$ with a triangular density. To get a uniform distribution, the variables or their sum must be scaled or transformed accordingly.

How does the convolution of two uniform distributions relate to the sum of the variables?

The convolution of two $\text{uniform}(0,1)$ distributions results in a triangular distribution over $[0,2]$, which describes the probability density function of their sum.

Is there a scenario where the sum of two $\text{uniform}(0,1)$ variables is uniformly distributed over a sub-interval?

While the sum over $[0,2]$ is triangular, by applying specific transformations or restrictions, one can create scenarios where a scaled or shifted sum is uniformly distributed over a sub-interval, but the raw sum itself is not.

What practical applications are related to the distribution of the sum of uniform random variables?

Understanding the sum's distribution is useful in areas like simulation, modeling of combined uncertainties, and in designing algorithms that rely on the aggregation of uniform random inputs.

How would you simulate the sum of two $\text{uniform}(0,1)$ variables in a programming language?

Generate two independent uniform random variables using a random number generator (e.g., `random.uniform(0,1)` in Python) and sum them. The resulting distribution will approximate the triangular shape over $[0,2]$ with enough samples.

Additional Resources

The Sum Of Two Independent Uniformly Distributed Random Variables Over $[0,1]$ Is Another Uniformly Distributed

In the realm of probability theory and statistics, understanding the behavior of random variables under various operations is fundamental. One particularly elegant result involves the sum of two independent uniform variables on the same interval: it turns out that their sum is also uniformly distributed, albeit over a different range. This surprising and beautiful property has implications across fields—from statistical modeling to signal processing—and offers insight into the nature of randomness itself. In this article, we will explore this phenomenon in depth, unpack the mathematical reasoning behind it, and discuss its significance.

Understanding the Uniform Distribution on $[0,1]$

Before diving into the sum of two such variables, it's essential to understand what a uniform distribution over the interval $[0,1]$ entails.

Definition and Properties

- A random variable (X) is said to be uniformly distributed over $[0,1]$, denoted as $(X \sim U(0,1))$, if every subinterval within $[0,1]$ has an equal probability proportional to its length.
- The probability density function (pdf) of (X) is constant:

$$f_X(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

- The cumulative distribution function (CDF) is:

$$F_X(x) = \begin{cases} 0 & x < 0, \\ x & 0 \leq x \leq 1, \\ 1 & x > 1. \end{cases}$$

Independence

- Two random variables (X) and (Y) are independent if the joint distribution factors into the product of their marginal distributions:

$$f_{\{X,Y\}}(x,y) = f_X(x) \times f_Y(y).$$

- For two independent $(U(0,1))$ variables, their joint pdf over the unit square $[0,1] \times [0,1]$ is constant:

$$f_{\{X,Y\}}(x,y) = 1, \quad \text{for } 0 \leq x,y \leq 1.$$

The Sum of Two Independent Uniform(0,1) Variables

Let (X) and (Y) be independent and identically distributed as $(U(0,1))$. Define a new random variable:

$$Z = X + Y.$$

The question that intrigues statisticians and mathematicians alike is: What is the distribution of (Z) ?

Initial Intuition

- Since both (X) and (Y) are bounded between 0 and 1, their sum (Z) must lie within $[0, 2]$.
- The distribution of (Z) isn't uniform over $[0, 2]$. Instead, it exhibits a particular shape, often described as a "triangular" distribution, which we will explore.

Deriving the Distribution of (Z)

To understand the distribution of (Z) , we turn to its probability density function (pdf). The key tool here is convolution, which combines the distributions of independent random variables.

Mathematical Derivation

The pdf of $(Z = X + Y)$ is given by the convolution of (f_X) and (f_Y) :

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(t) f_Y(z - t) dt.$$

Given the uniform PDFs:

$$f_X(t) = 1 \quad \text{for } 0 \leq t \leq 1,$$
$$f_Y(z - t) = 1 \quad \text{for } 0 \leq z - t \leq 1,$$

and zero elsewhere. Therefore, the limits of integration depend on (z) :

- For $(z \in [0, 1])$:

The bounds are from $(t=0)$ to $(t=z)$.

- For $(z \in [1, 2])$:

The bounds are from $(t=z-1)$ to $(t=1)$.

Putting this together:

$$f_Z(z) = \begin{cases} \int_0^z 1 \, dt = z, & 0 \leq z \leq 1, \\ \int_{z-1}^1 1 \, dt = 2 - z, & 1 < z \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Resulting Distribution: Triangular Shape

The pdf of Z is:

$$f_Z(z) = \begin{cases} z, & 0 \leq z \leq 1, \\ 2 - z, & 1 < z \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

This is a triangular distribution—a classic shape rising linearly from 0 to 1, then decreasing linearly back to 0 over $[1, 2]$.

Is the Sum Uniformly Distributed?

The core question: Is $Z = X + Y$ itself uniformly distributed over $[0, 2]$?

Answer: No.

The distribution derived above is not uniform; it is triangular. The pdf is not constant but peaks at $z=1$, the midpoint of the interval.

So, where does the statement that the sum is uniformly distributed come from?

This is a common misconception stemming from a misinterpretation of the properties of uniform variables or from specific conditions under which the sum behaves differently. The key is to understand that:

- The sum of two independent uniform variables over $[0, 1]$ is not uniformly distributed over $[0, 2]$.
- Instead, it follows a triangular distribution, as shown.

The Special Case: Discrete Uniform Variables and Other Operations

While the sum of two independent uniform(0,1) variables is not uniform over $[0, 2]$, other operations or different distributions can exhibit surprising properties.

For example:

- The sum of two discrete uniform variables over finite sets can sometimes produce uniform distributions over their sum, under specific conditions.
- The sum of two independent uniform variables over different intervals, or with different parameters, can lead to other well-understood distributions.
- Certain transformations or linear combinations may preserve uniformity under specific circumstances.

Key Point: The statement that the sum of two independent uniform(0,1) variables is another uniform is not true in general. The correct distribution is triangular, not uniform.

Clarifying the Misconception

The confusion may arise from the following:

- The product of two independent uniform variables over [0,1] is not uniform, but the distribution of the minimum or maximum can have uniform-like properties in some contexts.
- When considering conditional distributions or specific transformations, sometimes the resulting distribution appears uniform, but these are special cases.

In summary:

Operation	Resulting Distribution	Is it uniform?
Sum of two independent $U(0,1)$	Triangular over [0,2]	No
Product of two independent $U(0,1)$	Beta distribution with parameters (1,1) scaled	No
Difference of two independent $U(0,1)$	Triangular over [-1,1]	No

Broader Implications and Applications

Understanding the distribution of the sum of independent uniform variables is fundamental in various applications:

- Simulation and Monte Carlo methods: Knowing the resulting distribution helps in designing algorithms that rely on sum operations.
- Signal processing: When combining signals with uniform noise, the resulting combined noise distribution can be triangular.
- Statistical modeling: Recognizing the shape of the sum's distribution aids in hypothesis testing and model fitting.
- Probability theory education: Demonstrates how intuitive assumptions about distributions can be misleading without rigorous derivation.

Furthermore, this result highlights the importance of convolution as a tool in probability theory, enabling us to derive the distribution of sums of independent variables systematically.

Conclusion

While the statement "The sum of two independent uniform variables over [0,1] is another uniform" might sound elegant and straightforward, the reality is more nuanced. The sum is not uniform but follows a well-known triangular distribution over [0,2]. This property underscores the richness of probability distributions and the necessity of rigorous derivation in understanding their behavior.

Key takeaways:

- The sum of two independent $U(0,1)$ variables is triangular over $[0,2]$, not uniform.
- Convolution provides a systematic way to derive the distribution of sums.
- Recognizing misconceptions facilitates better modeling and analysis in practical applications.
- The interplay of distribution operations reveals the intricate structure underlying randomness.

In essence, this exploration exemplifies how even simple-looking operations—adding two uniform random variables—unveil complex and elegant probabilistic structures, enriching our understanding of randomness and variability.

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