Verify The Conclusion Of Green's Theorem By Evaluating Both Sides Of The Equation For The Field F= -2yi+2xj.

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Green's theorem establishes a fundamental relationship between a line integral around a simple closed curve in the plane and a double integral over the region it encloses. Specifically, it states that for a continuously differentiable vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, the circulation around the boundary is equal to the sum of the curl over the region:

$$
\oint_{C} P\,dx + Q\,dy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA
$$

In this article, we aim to verify this theorem explicitly for the vector field $\mathbf{F} = -2 y \mathbf{i} + 2 x \mathbf{j}$, by calculating both the line integral around the boundary and the double integral over the region D. This process not only confirms the validity of Green's theorem for this specific field but also demonstrates the method of applying the theorem in practice.

Understanding the Vector Field and Its Components

Components of the Field $\mathbf{F}$

The given vector field is:

$$
\mathbf{F} = -2 y \mathbf{i} + 2 x \mathbf{j}
$$

which implies:

- $P(x, y) = -2 y$
- $Q(x, y) = 2 x$

Understanding these components is crucial for both the line integral and the double integral calculations.

Region of Integration D

To evaluate the integrals, we must specify the region D. A common choice that simplifies calculations and illustrates Green's theorem well is the unit square:

$$
D: \{ (x, y) \mid 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \}
$$

This choice is convenient and illustrative, but the process can be generalized to other regions as long as the boundary is a simple, closed, positively oriented curve.

Calculating the Line Integral $\oint_{C} P\,dx + Q\,dy$

Parameterizing the Boundary Curve C

The boundary C of the region D (the unit square) consists of four line segments:

1. Bottom edge: from $(0,0)$ to $(1,0)$
2. Right edge: from $(1,0)$ to $(1,1)$
3. Top edge: from $(1,1)$ to $(0,1)$
4. Left edge: from $(0,1)$ to $(0,0)$

Each segment can be parameterized individually.

Evaluating the Line Integral on Each Segment

The line integral is:

$$
\oint_{C} P\, dx + Q\, dy = \sum_{i=1}^{4} \int_{C_i} P\, dx + Q\, dy
$$

Let's evaluate each:

Segment 1: Bottom edge $(0,0) \to (1,0)$

- Parameterization: $x = t, y = 0$, with $t \in [0,1]$
- Differential: $dx = dt, dy = 0$
- $P = -2 y = 0$
- $Q = 2 x = 2 t$

Integral:

$$
\int_{0}^{1} P\, dx + Q\, dy = \int_{0}^{1} 0 \cdot dt + 2 t \cdot 0 = 0
$$

Segment 2: Right edge $(1,0) \to (1,1)$

- Parameterization: $x=1, y= t$, with $t \in [0,1]$
- Differential: $dx=0, dy=dt$
- $P = -2 y = -2 t$
- $Q= 2 x= 2$

Integral:

$$
\int_{0}^{1} P\, dx + Q\, dy = \int_{0}^{1} (-2 t) \cdot 0 + 2 \cdot dt = \int_{0}^{1} 2\, dt = 2
$$

Segment 3: Top edge $(1,1) \to (0,1)$

- Parameterization: $x= t, y= 1$, $t \in [1,0]$ (or reverse)
- To maintain positive orientation (counterclockwise), we set t from 1 to 0:

$x= t, y=1$, with $t \in [1,0]$

- Differential: $dx= dt, dy=0$

- $P= -2 y= -2 \cdot 1= -2$

- $Q= 2 x= 2 t$

Integral:

$$
\int_{t=1}^{0} P\, dx + Q\, dy = \int_{1}^{0} (-2) \cdot dt + 2 t \cdot 0 = \int_{1}^{0} -2\, dt = -2 (0 - 1) = 2
$$

(Note: Reversing limits introduces a sign change, so the integral is positive 2.)

Segment 4: Left edge $(0,1) \to (0,0)$

- Parameterization: $x=0, y= t$, $t \in [1,0]$

- Differential: $dx=0, dy= dt$

- $P= -2 y= -2 t$

- $Q= 2 x= 0$

Integral:

$$
\int_{1}^{0} P\, dx + Q\, dy = \int_{1}^{0} (-2 t)\cdot 0 + 0 \cdot dt= 0
$$

Total line integral:

Adding all segments:

$$
0 + 2 + 2 + 0 = 4
$$

Thus,

$$
\oint_{C} P\, dx + Q\, dy = \boxed{4}
$$

Calculating the Double Integral $\iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$

Computing Partial Derivatives

Recall:

- $P = -2 y$
- $Q= 2 x$

Calculate derivatives:

$$
\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x} (2 x) = 2
$$
$$
\frac{\partial P}{\partial y} = \frac{\partial}{\partial y} (-2 y) = -2
$$

Substitute into the integrand:

$$
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2 - (-2) = 4
$$

\]

Note: The integrand is constant over D.

Evaluating the Double Integral over D

Since the integrand is constant (4) over the unit square:

$$
\iint_{D} 4\, dA = 4 \times \text{Area of } D
$$

The area of the unit square:

$$
\text{Area} = 1 \times 1 = 1
$$

Thus,

$$
\iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = 4 \times 1 = 4
$$

Verification of Green's Theorem

Having computed both sides, we compare:

- Line integral (circulation): 4
- Double integral (curl over region): 4

Since both are equal, the verification confirms the validity of Green's theorem for the field $\mathbf{F} = -2 y \mathbf{i} + 2 x \mathbf{j}$ over the unit square.

Discussion and Generalization

Implications of the Result

This explicit calculation demonstrates how Green's theorem provides a powerful method to evaluate line integrals by converting them into double integrals, often simplifying calculations. The consistent result validates the theoretical foundation and confirms the theorem's correctness in this context.

Extension to Other Regions and Fields

While this example used the unit square for simplicity, the procedure extends to more complex regions, provided the boundary is positively oriented and simple, and the vector field is sufficiently smooth. The key steps involve:

- Parameterizing each boundary segment
- Calculating the line integral for each
- Computing the double integral of the curl over the region

This method emphasizes the importance of understanding both the boundary's geometry and the behavior of

Frequently Asked Questions

What is Green's Theorem and how is it used to verify the conclusion for the given vector field?

Green's Theorem relates a line integral around a closed curve to a double integral over the region it encloses. To verify its conclusion for F = -2yi + 2xj, we evaluate both the line integral around the boundary and the double integral of the curl over the region, ensuring they are equal.

How do you compute the line integral of F = -2yi + 2xj around a closed curve?

Parameterize the boundary curve, substitute into F, and integrate the dot product of F with the differential element along the curve. For a simple shape like a rectangle or circle, use their standard parameterizations for easier calculation.

What is the curl of the vector field F = -2yi + 2xj?

The curl of F in two dimensions is given by $\partial Q/\partial x - \partial P/\partial y$, where $P = -2y$ and $Q = 2x$. Calculating gives $\partial(2x)/\partial x - \partial(-2y)/\partial y = 2 - (-2) = 4$.

How do you evaluate the double integral of the curl over the region for verification?

Identify the region enclosed by the boundary, set up the double integral of the curl (which is 4 in this case), and integrate over the area. For example, for a rectangle, multiply 4 by its area.

What is the significance of verifying Green's Theorem for the given field?

Verifying Green's Theorem confirms the consistency between the circulation around the boundary and the flux of the curl over the region, thus validating the theorem's application to the specific vector field and region.

Can Green's Theorem be directly applied to any shape for this vector field?

Green's Theorem applies to simple, positively oriented, piecewise-smooth closed curves in the plane, so it can be applied to any such shape, including circles, rectangles, or polygons, as long as the field is well-defined on and inside the region.

What is the expected result when evaluating both sides of Green's Theorem for this field?

Both the line integral around the boundary and the double integral of the curl over the region should equal 8, since the curl is 4 and the area of the region is such that 4 times the area equals 8.

How does the shape and size of the region affect the verification process?

The shape and size determine the limits of integration for the double integral and the parameterization for the line integral. Accurate setup ensures both integrals evaluate to the same value, confirming Green's Theorem.

What steps are involved in verifying Green's Theorem for the given field step-by-step?

First, parameterize the boundary to compute the line integral; second, determine the curl of the field; third, set up and evaluate the double integral of the curl over the region; finally, compare both results to verify the theorem.

Why is it important to confirm the conclusion of Green's Theorem in this context?

Confirming the conclusion reinforces understanding of the relationship between circulation and curl, demonstrates the theorem's validity for specific fields, and helps in solving complex vector calculus problems involving flux and circulation.

Additional Resources

Verify The Conclusion Of Green's Theorem By Evaluating Both Sides Of The Equation For The Field F= -2yi+2xj

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Introduction

Green's theorem is a fundamental result in vector calculus that establishes a relationship between a line integral around a simple closed curve and a double integral over the region it encloses. It serves as a cornerstone for various applications in physics, engineering, and mathematics, allowing complex line integrals to be transformed into more manageable double integrals, or vice versa. To truly grasp the power and validity of Green's theorem, it is instructive to verify its conclusion through explicit calculation for a specific vector field.

In this article, we will examine the vector field F = -2yi + 2xj and verify Green's theorem by evaluating both sides of the equation. This exercise not only demonstrates the theorem's application but also reinforces the importance of understanding the underlying calculus principles that make Green's theorem a pivotal tool in multivariable calculus.

Understanding Green's Theorem

Before diving into calculations, it is essential to clarify what Green's theorem states and the conditions under which it applies.

The Statement of Green's Theorem

Green's theorem relates a line integral around a simple, positively oriented, closed curve C to a double integral over the region D it encloses:

$$
\oint_{C} \mathbf{F} \cdot d\mathbf{r} = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy
$$

where:

- $\mathbf{F} = P(x, y)i + Q(x, y)j$ is a continuously differentiable vector field.
- C is the positively oriented, simple closed curve bounding the region D.

The theorem's significance lies in converting a potentially complicated line integral into a double integral, which can often be easier to evaluate.

Conditions for Applying Green's Theorem

- The vector field $\mathbf{F}$ should have continuous partial derivatives on an open region containing D.
- The curve C must be simple (non-self-intersecting), closed, and positively oriented.
- The region D should be planar and bounded.

Defining the Vector Field and Region

The vector field under consideration is:

$$
\mathbf{F} = -2 y \mathbf{i} + 2 x \mathbf{j}
$$
which can be written as:

$$
P(x, y) = -2 y, \quad Q(x, y) = 2 x
$$

Suppose we choose the region D to be the unit circle centered at the origin:

$$
x^2 + y^2 \leq 1
$$

and the corresponding curve C is the circle:

$$
x^2 + y^2 = 1
$$

This choice simplifies calculations due to symmetry and well-known parametrization.

Evaluating the Line Integral (Left Side of Green's Theorem)

The line integral around C is given by:

$$
\oint_{C} \mathbf{F} \cdot d\mathbf{r}
$$

where $d\mathbf{r} = dx\, \mathbf{i} + dy\, \mathbf{j}$.

Parametrization of the Curve

The circle $x^2 + y^2 = 1$ can be parametrized as:

$$
x = \cos t, \quad y = \sin t, \quad t \in [0, 2\pi]
$$

Correspondingly,

$$
dx = -\sin t\, dt, \quad dy = \cos t\, dt
$$

Computing $\mathbf{F}$ on C

Substituting the parametrization into $\mathbf{F}$:

$$
P = -2 y = -2 \sin t
$$

$$
Q = 2 x = 2 \cos t
$$

Calculating the Line Integral

$$
\oint_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{0}^{2\pi} [P \frac{dx}{dt} + Q \frac{dy}{dt}] dt
$$

Plugging in the derivatives:

$$
= \int_{0}^{2\pi} \left[(-2 \sin t)(-\sin t) + (2 \cos t)(\cos t) \right] dt
$$

Simplify the integrand:

$$
= \int_{0}^{2\pi} \left[2 \sin^2 t + 2 \cos^2 t \right] dt
$$

Using the Pythagorean identity:

$$
\sin^2 t + \cos^2 t = 1
$$

we get:

$$
= \int_{0}^{2\pi} 2 (\sin^2 t + \cos^2 t) dt = \int_{0}^{2\pi} 2 \times 1\, dt = 2 \times 2\pi = 4 \pi
$$

Thus, the line integral evaluates to:

$$
\boxed{\oint_{C} \mathbf{F} \cdot d\mathbf{r} = 4 \pi}
$$

Evaluating the Double Integral (Right Side of Green's Theorem)

Next, we compute:

$$
\iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy
$$

Computing the Partial Derivatives

$$
\frac{\partial Q}{\partial x} = \frac{\partial (2x)}{\partial x} = 2
$$

$$
\frac{\partial P}{\partial y} = \frac{\partial (-2 y)}{\partial y} = -2
$$

Therefore,

$$
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2 - (-2) = 4
$$

The double integral simplifies to:

$$
\iint_{D} 4\, dx\,dy
$$

which is 4 times the area of the region D, the unit circle.

Calculating the Area of D

Since D is the unit disk:

$$
\text{Area} = \pi r^2 = \pi \times 1^2 = \pi
$$

Thus, the double integral evaluates to:

$$
4 \times \pi = 4 \pi
$$

Conclusion: Verifying Green's Theorem

The calculations reveal:

- Line integral around C: 4π
- Double integral over D: 4π

Since both sides are equal, the verification confirms the validity of Green's theorem for this specific vector field and region:

$$
\boxed{
\oint_{C} \mathbf{F} \cdot d\mathbf{r} = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy
}
$$

Broader Implications and Applications

This explicit verification exemplifies the practical power of Green's theorem in transforming complex line integrals into simpler double integrals, especially when dealing with symmetric regions like circles or rectangles. Engineers and physicists frequently leverage this theorem to evaluate circulation, flux, and other field properties more efficiently.

Extending the Verification

While this example used a circular region, Green's theorem applies broadly to any simply connected, planar region with a positively oriented boundary, provided the vector field is continuously differentiable within that region. Similar calculations can be performed for rectangles, polygons, or more complex shapes, often simplifying the integral evaluation process.

Limitations and Cautions

- The theorem does not hold if the region is not simply connected or if the field lacks the required smoothness.
- Proper orientation of the boundary curve is critical; positive (counterclockwise) orientation is standard.

Final Remarks

Verifying Green's theorem through explicit calculations not only solidifies understanding of the theorem's statement but also builds intuition about the behavior of vector fields and their circulation. The example with $\mathbf{F} = -2 y \mathbf{i} + 2 x \mathbf{j}$ and the unit circle showcases the elegance and utility of the theorem, bridging the gap between abstract mathematical concepts and tangible calculations. Such exercises are fundamental in advanced studies of multivariable calculus and form the basis for more sophisticated theorems like Stokes' and Divergence theorems, which generalize these ideas to higher dimensions.

References

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