3. The Function F(x)=sqrt(3)sinx+cosx,0x And Function G(x)=2x,xR (a) Find (fg)(x). (b) Solve (fg)(x)=2cos2x,0x.

3. The Function F(x)=sqrt(3)sinx + cosx, 0 < x < π And Function G(x)=2x, x ∈ R (a) Find (fg)(x). (b) Solve (fg)(x) = 2cos2x, 0 < x < π.

Understanding the behavior of functions and their compositions is a fundamental aspect of mathematical analysis. In this article, we explore two specific functions:

- $F(x) = \sqrt{3} \sin x + \cos x$, where $0 < x < \pi$
- $G(x) = 2x$, where $x \in R$

Our goal is to perform the composition $(f \circ g)(x) = f(g(x))$, find its explicit form, and then solve the equation $(f \circ g)(x) = 2 \cos 2x$ within the given domain. These tasks will enhance your understanding of function composition, trigonometric identities, and solving trigonometric equations.

Understanding the Functions

Before delving into the calculations, let’s examine the individual functions and their properties.

Function F(x): A Linear Combination of Trigonometric Functions

- $F(x) = \sqrt{3} \sin x + \cos x$

This function is a linear combination of sine and cosine functions. Its structure suggests that it can be rewritten using a single sinusoid with a phase shift, using the method of combining sinusoidal functions:

$$ a \sin x + b \cos x = R \sin(x + \phi) $$

where

$$ R = \sqrt{a^2 + b^2} $$
$$ \phi = \arctan\left(\frac{b}{a}\right) $$

In our case:

- $a = \sqrt{3}$
- $b = 1$

Calculating R:

$$ R = \sqrt{ (\sqrt{3})^2 + 1^2 } = \sqrt{ 3 + 1 } = \sqrt{4} = 2 $$

Calculating phase shift ϕ:

$$ \phi = \arctan\left(\frac{1}{\sqrt{3}}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} $$

Therefore,

$$ F(x) = 2 \sin(x + \frac{\pi}{6}) $$

This form simplifies understanding the behavior of F(x) over its domain.

Function G(x): A Linear Function

- G(x) = 2x

This is a simple linear function defined for all real numbers. Its straightforward nature makes it easy to analyze when composing with other functions.

Part (a): Find $(f \circ g)(x) = f(g(x))$

Given the functions:

- $f(x) = \sqrt{3} \sin x + \cos x$
- $g(x) = 2x$

The composition $(f \circ g)(x)$ is:

$$ (f \circ g)(x) = f(g(x)) = f(2x) $$

Substituting $2x$ into $f(x)$:

$$ (f \circ g)(x) = \sqrt{3} \sin(2x) + \cos(2x) $$

This expression involves the double-angle identities for sine and cosine, which are:

- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x$

But since the expression is already in terms of $\sin(2x)$ and $\cos(2x)$, it's often more useful to analyze or simplify it further using identities or phase shift methods.

Expressing $(f \circ g)(x)$ as a Single Sinusoid

Recall that any linear combination of sine and cosine with the same argument can be written as a single sinusoid:

$$ A \sin \theta + B \cos \theta = R \sin(\theta + \phi) $$

where

$$ R = \sqrt{A^2 + B^2} $$
$$ \phi = \arctan\left(\frac{B}{A}\right) $$

Applying this to:

$$ (f \circ g)(x) = \sqrt{3} \sin(2x) + \cos(2x) $$

Set:

- $A = \sqrt{3}$
- $B = 1$

Calculate R:

$$ R = \sqrt{ (\sqrt{3})^2 + 1^2 } = \sqrt{3 + 1} = 2 $$

Calculate phase shift ϕ:

$$ \phi = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} $$

Thus,

$$ (f \circ g)(x) = 2 \sin\left(2x + \frac{\pi}{6}\right) $$

Final expression:

$$ \boxed{
(f \circ g)(x) = 2 \sin\left(2x + \frac{\pi}{6}\right)
} $$

This form simplifies the analysis of the composition, especially when solving equations involving $(f \circ g)(x)$.

Part (b): Solve $(f \circ g)(x) = 2 \cos 2x$, $0 < x < \pi$

We need to solve:

$$ 2 \sin\left(2x + \frac{\pi}{6}\right) = 2 \cos 2x $$

Dividing both sides by 2:

$$ \sin\left(2x + \frac{\pi}{6}\right) = \cos 2x $$

Recall the identity:

$$ \cos \alpha = \sin\left(\frac{\pi}{2} - \alpha\right) $$

Using this, rewrite the right side:

$$ \sin\left(2x + \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{2} - 2x\right) $$

So, the equation is:

$$ \sin\left(2x + \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{2} - 2x\right) $$

Solution of the sine equation:

The general solutions for:

$$ \sin A = \sin B $$

are:

$$ A = B + 2k\pi \quad \text{or} \quad A = \pi - B + 2k\pi, \quad k \in \mathbb{Z} $$

Set:

- $A = 2x + \frac{\pi}{6}$
- $B = \frac{\pi}{2} - 2x$

Case 1:

$$ 2x + \frac{\pi}{6} = \frac{\pi}{2} - 2x + 2k\pi $$

Solve for x:

$$ 2x + 2x = \frac{\pi}{2} - \frac{\pi}{6} + 2k\pi $$

$$ 4x = \left(\frac{3\pi}{6} - \frac{\pi}{6} \right) + 2k\pi $$

$$ 4x = \frac{2\pi}{6} + 2k\pi = \frac{\pi}{3} + 2k\pi $$

$$ x = \frac{\pi}{12} + \frac{k\pi}{2} $$

Case 2:

$$ 2x + \frac{\pi}{6} = \pi - \left(\frac{\pi}{2} - 2x \right) + 2k\pi $$

Simplify the right side:

$$ \pi - \frac{\pi}{2} + 2x + 2k\pi = \frac{\pi}{2} + 2x + 2k\pi $$

Set equal:

$$ 2x + \frac{\pi}{6} = \frac{\pi}{2} + 2x + 2k\pi $$

Subtract $2x$ from both sides:

$$ \frac{\pi}{6} = \frac{\pi}{2} + 2k\pi $$

Solve for k:

$$ \frac{\pi}{6} - \frac{\pi}{2} = 2k\pi $$

Express both with common denominator:

$$ \frac{\pi}{6} - \frac{3\pi}{6} = 2k\pi $$

$$ -\frac{2\pi}{6} = 2k\pi $$

$$ -\frac{\pi}{3} = 2k\pi $$

Divide both sides by 2π:

$$ -\frac{1}{6} = k $$

Since k must be an integer, this is not valid. Therefore, no solutions arise from case 2.

Final Solution Set for x in (0, π)