

Consider The Vector Space $\mathbb{R}^2$ And Two Sets Of Vectors $S = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (vertical) $S' = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (vertical)(a)

Consider The Vector Space $\mathbb{R}^2$ And Two Sets Of Vectors $S = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (vertical) $S' = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (vertical)(a). Exploring the properties of these vector sets within the two-dimensional real vector space ($\mathbb{R}^2$) provides valuable insights into fundamental concepts such as span, linear independence, basis, subspaces, and linear transformations. Understanding how these sets interact and relate to the structure of $\mathbb{R}^2$ is essential for students and professionals working in linear algebra, computer graphics, data science, and related fields. This article delves into a comprehensive analysis of the sets S and S' , examining their properties, relationships, and significance in the broader context of vector space theory.

Understanding the Vector Space $\mathbb{R}^2$

The two-dimensional real vector space, $\mathbb{R}^2$, consists of all ordered pairs of real numbers (x, y) . It is a fundamental construct in linear algebra, representing geometric entities like points, lines, and planes in a coordinate system.

Key Features of $\mathbb{R}^2$:

- Vectors: Elements of $\mathbb{R}^2$, e.g., (x, y) .
- Operations: Vector addition and scalar multiplication.
- Subspaces: Lines through the origin, planes, and the entire space itself.

The study of vectors in $\mathbb{R}^2$ involves analyzing their linear relationships, spans, and how they can form bases for the space.

Analyzing the Sets S and S'

Given the two sets:

- $S = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}^T, \begin{bmatrix} 1 \\ 2 \end{bmatrix}^T \right\}$
- $S' = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T, \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \right\}$

where the notation $\begin{bmatrix} a & b \end{bmatrix}^T$ indicates a column vector with components a and b .

Objective:

- Determine if the vectors in each set are linearly independent.
- Find the span of each set.
- Assess whether these sets form bases for $\mathbb{R}^2$.
- Explore relationships between S and S' .

Linear Independence and Dependence

A set of vectors is linearly independent if no vector can be expressed as a linear combination of the others. Conversely, if such an expression exists, the vectors are linearly dependent.

For Set S:

Vectors:

$$- v_1 = [2 \ 1]^T$$

$$- v_2 = [1 \ 2]^T$$

To check independence:

- Solve for scalars a, b such that $av_1 + bv_2 = 0$ vector.

Equation:

$$a [2 \ 1]^T + b [1 \ 2]^T = [0 \ 0]^T$$

Component-wise:

$$- 2a + b = 0$$

$$- a + 2b = 0$$

Solve:

$$\text{From first: } b = -2a$$

Substitute into second:

$$a + 2(-2a) = 0 \rightarrow a - 4a = 0 \rightarrow -3a = 0 \rightarrow a = 0$$

$$\text{Then } b = -2(0) = 0$$

Since the only solution is $a = 0, b = 0$, vectors are linearly independent.

For Set S':

Vectors:

$$- v_1' = [1 \ 0]^T$$

$$- v_2' = [1 \ 1]^T$$

Check for scalars a, b :

$$a [1 \ 0]^T + b [1 \ 1]^T = [0 \ 0]^T$$

Component-wise:

$$- a + b = 0$$

$$- 0 + b = 0$$

From second:

$$b = 0$$

From first:

$$a + 0 = 0 \rightarrow a = 0$$

Again, only trivial solution, so vectors are linearly independent.

Span and Basis

The span of a set of vectors is all linear combinations of those vectors. If the span covers the entire $\mathbb{R}^2$, the set forms a basis for the space.

Span of S:

- Since v_1 and v_2 are linearly independent in $\mathbb{R}^2$, their span is a 2-dimensional subspace—specifically, the entire $\mathbb{R}^2$.
- Conclusion: S is a basis for $\mathbb{R}^2$.

Span of S':

- Similarly, v_1' and v_2' are linearly independent.
- Their span is $\mathbb{R}^2$.
- Conclusion: S' is also a basis for $\mathbb{R}^2$.

Comparing the Sets S and S'

Both sets are bases for $\mathbb{R}^2$, but their vectors are different, leading to various geometric and algebraic interpretations.

Geometric Interpretation

- Set S: Vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}^T$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}^T$ form a basis with directions along certain lines in the plane.
- Set S': Vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}^T$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}^T$ form a basis aligned with coordinate axes and diagonals.

Visualizing the vectors:

- S vectors are inclined in such a way that they span the entire plane.
- S' vectors are aligned along the x-axis and the line $y = x$.

Change of Basis and Coordinate Transformation

Since both sets form bases, they can be used to define coordinate systems within $\mathbb{R}^2$.

Transformation matrix from S to S':

- Construct matrices with vectors as columns:

- For S:

$$(M_S = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix})$$

- For S':

$$(M_{S'} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix})$$

- The change of basis matrix from S to S' is:

$$(P = M_{S'}^{-1} M_S)$$

Calculating $(M_{S'}^{-1})$:

- Determinant: $(1)(1) - (1)(0) = 1$

- Inverse:

$$(M_{S'}^{-1}) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Then:

$$(P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} (1)(2) + (-1)(1) & (1)(1) + (-1)(2) \\ (0)(2) + (1)(1) & (0)(1) + (1)(2) \end{bmatrix} = \begin{bmatrix} 2 - 1 & 1 - 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix})$$

This matrix describes how to convert coordinates from one basis to another.

Applications and Significance in Linear Algebra

Understanding the properties of sets S and S' provides foundational knowledge for many applications.

1. Basis and Coordinate Systems

- Both sets serve as bases for $\mathbb{R}^2$, allowing representation of any vector as a linear combination.
- Facilitates coordinate transformations and simplifies vector calculations.

2. Subspace Analysis

- Both sets span the entire space, illustrating the concept of bases and their role in decomposing vector spaces.

3. Linear Transformations

- Change of basis matrices enable the study of linear transformations and their matrix representations.
- Essential in computer graphics, robotics, and physics.

4. Solving Systems of Linear Equations

- The invertibility of matrices associated with these sets guarantees unique solutions to linear systems.

Conclusion

The sets $S = \{[2 \ 1], [1 \ 2]\}$ and $S' = \{[1 \ 0], [1 \ 1]\}$ are both fundamental in understanding the structure of the vector space $\mathbb{R}^2$. They exemplify how different bases can span the entire space, providing flexibility in representing vectors and transforming between coordinate systems. Their linear independence ensures they are valid bases, and their respective spans highlight the richness of the geometric and algebraic relationships within $\mathbb{R}^2$. Whether applied in theoretical mathematics, computer graphics, or engineering, analyzing such vector sets deepens our comprehension of linear algebra's core principles, empowering practical problem-solving and advanced research.

Key Takeaways:

- Both S and S' are bases for $\mathbb{R}^2$.
- They are composed of linearly independent vectors.
- They facilitate coordinate transformations via change of basis matrices.
- Understanding these sets enhances comprehension of span, linear independence, and basis concepts in linear algebra.

By mastering the properties and applications of vector sets like S and S' , students and professionals can unlock powerful tools for analyzing complex systems and designing efficient algorithms across various scientific and engineering disciplines.

Frequently Asked Questions

Are the sets $S = \{[2, 1], [1, 2]\}$ and $T = \{[1, 0], [1, 1]\}$ in $\mathbb{R}^2$ linearly independent?

Yes, both sets consist of vectors that are linearly independent in $\mathbb{R}^2$, as none can be expressed as a scalar multiple or linear combination of the others within their respective sets.

Can the vectors in set $S = \{[2, 1], [1, 2]\}$ be used to span $\mathbb{R}^2$?

Yes, since the vectors are linearly independent in $\mathbb{R}^2$, they form a basis and can be used to span the entire $\mathbb{R}^2$ space.

Do the vectors in set $T = \{[1, 0], [1, 1]\}$ form a basis for $\mathbb{R}^2$?

Yes, the vectors are linearly independent and thus form a basis for $\mathbb{R}^2$, allowing any vector in $\mathbb{R}^2$ to be expressed as a linear combination of these two.

Are the sets S and T overlapping or related in terms of

span?

While both sets are linearly independent and form bases for $\mathbb{R}^2$, they span different subspaces; however, since both span $\mathbb{R}^2$, their spans are the same—covering the entire space.

Can vectors from set S be expressed as linear combinations of vectors from set T?

Yes, since both sets form bases of $\mathbb{R}^2$, any vector in S can be written as a linear combination of vectors in T, and vice versa.

What is the significance of these sets in understanding vector spaces in $\mathbb{R}^2$?

These sets illustrate how different collections of linearly independent vectors can serve as bases for $\mathbb{R}^2$, highlighting the flexibility in choosing bases and understanding span, independence, and dimension in vector spaces.

Additional Resources

Consider the vector space $\mathbb{R}^2$ and two sets of vectors $S = \{[2, 1], [1, 2]\}$ and $S' = \{[1, 0], [1, 1]\}$ —this scenario offers a compelling glimpse into the fundamental concepts of linear algebra, including spanning sets, linear independence, bases, and transformations within the two-dimensional Euclidean space. These two sets of vectors serve as excellent examples to understand how different configurations can either generate the entire space, form bases, or be dependent on each other. Exploring these sets not only deepens understanding of $\mathbb{R}^2$ but also highlights the importance of vector relationships in various applications, from computer graphics to data analysis.

Introduction to Vector Spaces and Sets of Vectors

What is the Vector Space $\mathbb{R}^2$?

The vector space $\mathbb{R}^2$, or the two-dimensional Euclidean space, consists of all ordered pairs of real numbers (x, y) . It is a foundational structure in linear algebra, serving as the setting for many geometric and algebraic concepts. Every vector in $\mathbb{R}^2$ can be visualized as a point or arrow originating from the origin.

The Role of Sets of Vectors

A set of vectors in $\mathbb{R}^2$ can serve various purposes:

- Spanning the space: whether the vectors can generate the entire $\mathbb{R}^2$ through linear combinations.
- Linearly independent: whether no vector in the set can be expressed as a linear combination of the others.

- Forming a basis: a minimal spanning set that is also linearly independent.

Understanding these properties helps determine how sets relate to the structure of $\mathbb{R}^2$, whether they can serve as coordinate systems, or whether they can be used to transform or analyze data.

Analyzing the Sets S and S' in $\mathbb{R}^2$

Given the sets:

- $S = \{[2, 1], [1, 2]\}$
- $S' = \{[1, 0], [1, 1]\}$

we will analyze their properties step-by-step.

Set S : $\{[2, 1], [1, 2]\}$

Visualizing the Vectors

- Vector $v_1 = [2, 1]$: points roughly towards the right and slightly upward.
- Vector $v_2 = [1, 2]$: points more vertically, but still to the right.

Are Vectors in S Linearly Independent?

To check, we examine whether one vector is a scalar multiple of the other:

- Is v_2 a scalar multiple of v_1 ? Let's see:

$$v_2 = \lambda v_1$$

$$\Rightarrow [1, 2] = \lambda [2, 1]$$

$$\Rightarrow 1 = 2\lambda \text{ and } 2 = 1\lambda$$

From the first: $\lambda = 1/2$

Plug into the second: $2 = 1 (1/2) = 0.5$, which is not equal to 2.

- Since no scalar λ satisfies both equations simultaneously, v_1 and v_2 are linearly independent.

Does S Span $\mathbb{R}^2$?

- Since the vectors are linearly independent and there are two of them in $\mathbb{R}^2$, they form a basis for $\mathbb{R}^2$.

- Implication: Any vector in $\mathbb{R}^2$ can be written as a linear combination of v_1 and v_2 .

Set S' : $\{[1, 0], [1, 1]\}$

Visualizing the Vectors

- Vector $v'_1 = [1, 0]$: points along the x-axis.
- Vector $v'_2 = [1, 1]$: points diagonally up and to the right.

Are Vectors in S' Linearly Independent?

Check whether v'_2 is a scalar multiple of v'_1 :

- Is $v'_2 = \lambda v'_1$?

$$\Rightarrow [1, 1] = \lambda [1, 0]$$

$$\Rightarrow 1 = \lambda \cdot 1 \Rightarrow \lambda = 1$$

$$\Rightarrow 1 = \lambda \cdot 0 \Rightarrow 1 = 0, \text{ which is false.}$$

- Therefore, v'_1 and v'_2 are linearly independent.

Does S' Span $\mathbb{R}^2$?

- Since the vectors are linearly independent and there are two in $\mathbb{R}^2$, they form a basis for $\mathbb{R}^2$ as well.

Comparing S and S' in Terms of Spanning and Bases

Both sets S and S' are bases of $\mathbb{R}^2$ because:

- They consist of two linearly independent vectors.
- They span the entire space.

Are S and S' Equivalent as Bases?

- Coordinate Systems: Each basis provides its own coordinate system for $\mathbb{R}^2$.
- Change of Basis: We can find the transition matrix between these bases to understand how vectors transform from one basis to the other.

Exploring the Relationships Between S and S'

Linear Combinations within Sets

- For S :

Any vector v in $\mathbb{R}^2$ can be written as:

$$v = a [2, 1] + b [1, 2]$$

- For S' :

Any vector v can be expressed as:

$$v = c [1, 0] + d [1, 1]$$

Transition Matrix from S' to S

To convert coordinates from the basis S' to S , we analyze how each vector in S' can be expressed in terms of S .

Expressing v'_1 in S :

- $v'_1 = [1, 0]$

- Find scalars α and β such that:

$$[1, 0] = \alpha [2, 1] + \beta [1, 2]$$

- Setting up equations:

$$1 = 2\alpha + 1\beta$$

$$0 = 1\alpha + 2\beta$$

- Solve the system:

From the second: $\alpha = -2\beta$

Substitute into the first:

$$1 = 2(-2\beta) + \beta = -4\beta + \beta = -3\beta$$

$$\Rightarrow \beta = -1/3$$

Then, $\alpha = -2(-1/3) = 2/3$

- Result:

$$v'_1 = (2/3) v_1 + (-1/3) v_2$$

Expressing v'_2 in S :

- $v'_2 = [1, 1]$

- Find α and β :

$$[1, 1] = \alpha [2, 1] + \beta [1, 2]$$

- Equations:

$$1 = 2\alpha + 1\beta$$

$$1 = 1\alpha + 2\beta$$

- Solve:

$$\text{From the second: } \alpha = 1 - 2\beta$$

Substitute into the first:

$$1 = 2(1 - 2\beta) + \beta = 2 - 4\beta + \beta = 2 - 3\beta$$

$$\Rightarrow 3\beta = 2 - 1 = 1$$

$$\Rightarrow \beta = 1/3$$

$$\alpha = 1 - 2(1/3) = 1 - 2/3 = 1/3$$

- Result:

$$v'_2 = (1/3) v_1 + (1/3) v_2$$

Transition Matrix from S' to S

Constructed with the coefficients:

$$\begin{array}{c|c|c} | & v'_1 & v'_2 \\ \hline | & v_1 & 2/3 & 1/3 \\ | & v_2 & -1/3 & 1/3 \end{array}$$

Expressed as a matrix:

$$\begin{pmatrix} 2/3 & 1/3 \\ -1/3 & 1/3 \end{pmatrix}$$

This matrix allows us to convert coordinates from the S' basis to the S basis.

Practical Implications and Applications

Coordinate Transformation

Understanding how to transition between different bases is crucial in many fields:

- Computer Graphics: transforming object coordinates from local to world space.
- Data Analysis: changing coordinate systems for principal component analysis.
- Engineering: analyzing forces and moments in different reference frames.

Linear Independence and Dependence

Knowing whether a set of vectors is linearly independent informs us about the minimal number of vectors needed to span a space, which impacts efficiency in computations and data representations.

Basis Selection

Choosing an appropriate basis can simplify problems, such as diagonalizing matrices or solving linear systems.

Summary of Key Takeaways

- Both S and S' are bases for $\mathbb{R}^2$, as their vectors are linearly independent and span the entire space.
- The vectors in S and S' are not scalar multiples of each other, but both form valid coordinate systems.
- Transition matrices enable conversion of coordinates between different bases, which is essential in many applied contexts.
- Linear algebra provides powerful tools to analyze and manipulate vector sets, with wide-ranging implications in science and engineering.

Final Thoughts

The exploration of the sets $S = \{[2, 1], [1, 2]\}$ and $S' = \{[1, 0], [1, 1]\}$ exemplifies core principles of linear algebra in $\mathbb{R}^2$. Recognizing whether vectors are linearly independent, whether they span the space, and how to transition between different bases equips students and practitioners with the foundational skills to analyze complex systems, optimize computations, and develop a deeper understanding of geometric and algebraic relationships.

By mastering these concepts, one gains the ability to navigate and manipulate the rich structure of

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