Find The Exact Value Of Tan, Given That Sin=1 /2and Is In Quadrant I. Rationalize Denominators When Applicable.

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Determining the exact value of tangent when given specific sine values is a fundamental concept in trigonometry. This problem not only reinforces the understanding of basic trigonometric ratios but also emphasizes the importance of quadrant considerations and the process of rationalizing denominators. In this article, we will explore how to find the exact value of tangent given that sine equals one-half and the angle is located in the first quadrant. We will go through the necessary steps, including referencing the Pythagorean identities, calculating cosine, and finally deriving the tangent value with proper rationalization.

Understanding the Basic Trigonometric Ratios

Before diving into calculations, it is crucial to recall the primary trigonometric functions and their relationships in a right-angled triangle.

Definitions of Sine, Cosine, and Tangent

- Sine ($\sin \theta$): Opposite side over hypotenuse
- Cosine ($\cos \theta$): Adjacent side over hypotenuse
- Tangent ($\tan \theta$): Opposite side over adjacent side

Given that $\sin \theta = \frac{1}{2}$, our goal is to find $\tan \theta$. Since the problem states that the angle is in the first quadrant, both sine and cosine are positive, simplifying the process.

Using the Pythagorean Identity to Find Cosine

The fundamental Pythagorean identity states:
$$
\sin^2 \theta + \cos^2 \theta = 1
$$

Given $\sin \theta = \frac{1}{2}$, substitute into the identity:
$$
\left(\frac{1}{2}\right)^2 + \cos^2 \theta = 1
$$

Simplify:

$$
\frac{1}{4} + \cos^2 \theta = 1
$$

Solve for $\cos^2 \theta$:

$$
\cos^2 \theta = 1 - \frac{1}{4} = \frac{3}{4}
$$

Now, take the square root:

$$
\cos \theta = \pm \sqrt{\frac{3}{4}} = \pm \frac{\sqrt{3}}{2}
$$

Since θ is in the first quadrant, where both sine and cosine are positive:

$$
\cos \theta = \frac{\sqrt{3}}{2}
$$

Summary:

$$
\sin \theta = \frac{1}{2} \quad \text{and} \quad \cos \theta = \frac{\sqrt{3}}{2}
$$

Calculating the Tangent

The tangent of an angle is the ratio of sine to cosine:

$$
\tan \theta = \frac{\sin \theta}{\cos \theta}
$$

Substitute the known values:

$$
\tan \theta = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}
$$

When dividing fractions, multiply the numerator by the reciprocal of the denominator:

$$
\tan \theta = \frac{1}{2} \times \frac{2}{\sqrt{3}} = \frac{1 \times 2}{2 \times \sqrt{3}} = \frac{2}{2 \sqrt{3}}
$$

Simplify the numerator and denominator:

$$
\tan \theta = \frac{1}{\sqrt{3}}
$$

Rationalizing the Denominator:
To express the tangent with a rational denominator, multiply numerator and denominator by $\sqrt{3}$:

$$
\tan \theta = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}
$$

Final exact value:

$$
\boxed{\tan \theta = \frac{\sqrt{3}}{3}}
$$

Summary of Steps

1. Identify the given information and the quadrant
- $\sin \theta = \frac{1}{2}$
- θ in the first quadrant (both sine and cosine are positive)

2. Use the Pythagorean identity to find cosine
- Calculate $\cos \theta = \frac{\sqrt{3}}{2}$

3. Compute tangent as the ratio of sine to cosine
- $\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$

4. Simplify the fraction and rationalize the denominator
- Result: $\frac{\sqrt{3}}{3}$

Additional Tips and Considerations

- Quadrant Significance: Remember that in the first quadrant, both sine and cosine are positive. In other quadrants, signs differ, affecting the tangent's sign.
- Rationalization: Always rationalize denominators when the problem specifies to do so, particularly for exact value expressions.
- Unit Circle Knowledge: Familiarity with the unit circle values for sine and cosine at common angles (30°, 45°, 60°) helps verify your answers quickly.

Conclusion

Determining the exact value of tangent given sine in the first quadrant involves understanding the relationship between the trigonometric functions, applying the Pythagorean identity, and simplifying fractions with rationalization. The key steps include calculating the cosine value from the sine, then forming the tangent ratio, and finally rationalizing the denominator for a clean, exact expression. This process reinforces fundamental principles of trigonometry, ensuring a solid foundation for more complex problems involving angles and their trigonometric ratios.

By practicing these steps, students and learners can confidently solve similar problems, enhance their understanding of the unit circle, and develop proficiency in handling rational expressions with radicals.

Frequently Asked Questions

Given that $\sin \theta = 1/2$ and θ is in Quadrant I, what is the exact value of $\tan \theta$?

Since $\sin \theta = 1/2$, $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - (1/2)^2} = \sqrt{1 - 1/4} = \sqrt{3/4} = \sqrt{3}/2$. Therefore, $\tan \theta = \sin \theta / \cos \theta = (1/2) / (\sqrt{3}/2) = (1/2)(2/\sqrt{3}) = 1/\sqrt{3}$. Rationalizing the denominator, $\tan \theta = (1/\sqrt{3})(\sqrt{3}/\sqrt{3}) = \sqrt{3}/3$.

Why is it valid to assume $\cos \theta$ is positive when θ is in Quadrant I?

In Quadrant I, both sine and cosine are positive, so $\cos \theta = \sqrt{1 - \sin^2 \theta}$ is positive, making the calculation valid without changing the sign.

How do you derive $\cos \theta$ from $\sin \theta$ for θ in Quadrant I?

Using the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$, substitute $\sin \theta = 1/2$: $\cos \theta = \sqrt{1 - (1/2)^2} = \sqrt{1 - 1/4} = \sqrt{3/4} = \sqrt{3}/2$.

What is the importance of rationalizing the denominator in the value of $\tan \theta$?

Rationalizing the denominator provides a simplified and standard form of the expression, making it easier to interpret and compare, especially in exact value calculations.

If $\sin \theta = 1/2$ and θ is in Quadrant I, what is the exact value of $\tan \theta$ without rationalization?

$\tan \theta = (1/2) / (\sqrt{3}/2) = (1/2)(2/\sqrt{3}) = 1/\sqrt{3}$.

Can the value of $\tan \theta$ be expressed in terms of sine or cosine only in this problem?

Yes, the tangent can be expressed as $\tan \theta = \sin \theta / \cos \theta$, which equals $(1/2) / (\sqrt{3}/2)$ before rationalization.

What is the final exact value of tan θ after rationalizing the denominator?

The exact value of tan θ is √3/3 after rationalizing the denominator.

Additional Resources

Find The Exact Value Of Tan, Given That Sin=1/2 and Is In Quadrant I

Understanding how to find the exact value of tangent when given the sine of an angle is a fundamental skill in trigonometry. The problem, "Find the exact value of tan, given that sin = 1/2 and the angle is in Quadrant I," is a classic example that tests your understanding of the relationships between the basic trigonometric functions. This type of problem emphasizes the importance of the Pythagorean identities and the geometric interpretation of sine and tangent within the unit circle. By carefully analyzing the given data and applying the appropriate formulas, you can determine the exact value of tangent and deepen your comprehension of trigonometric ratios.

Understanding the Problem

When approaching a trigonometry problem like this, it’s critical to understand what information is provided and what is being asked. The problem states:

- sin(θ) = 1/2
- θ is in Quadrant I

From this, we are asked to find tan(θ).

Since sine is defined as the ratio of the opposite side to the hypotenuse in a right triangle (or, equivalently, the y-coordinate of a point on the unit circle), and tangent is the ratio of sine to cosine (tan(θ) = sin(θ) / cos(θ)), our main task is to find the cosine value corresponding to the given sine.

Step-by-Step Solution Approach

1. Recognize the Pythagorean Identity

The primary identity relating sine and cosine is:

$$ \sin^2 \theta + \cos^2 \theta = 1 $$

Given that $\sin \theta = \frac{1}{2}$, we substitute this into the identity:

$$ \left(\frac{1}{2}\right)^2 + \cos^2 \theta = 1 $$

$$ \frac{1}{4} + \cos^2 \theta = 1 $$

$$ \cos^2 \theta = 1 - \frac{1}{4} = \frac{3}{4} $$

2. Find Cosine of the Angle

Taking the square root of both sides:

$$ \cos \theta = \pm \sqrt{\frac{3}{4}} = \pm \frac{\sqrt{3}}{2} $$

Since θ is in Quadrant I, where both sine and cosine are positive, we select the positive root:

$$ \cos \theta = \frac{\sqrt{3}}{2} $$

3. Calculate Tan(θ)

Recall that:

$$ \tan \theta = \frac{\sin \theta}{\cos \theta} $$

Substitute the known values:

$$ \tan \theta = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} $$

Simplify the complex fraction:

$$ \tan \theta = \frac{1/2}{\sqrt{3}/2} = \frac{1/2} \times \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}} $$

To write this rationalized:

$$ \tan \theta = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3} $$

Final answer:

$$\boxed{\tan \theta = \frac{\sqrt{3}}{3}}$$

Key Concepts and Tools Used

Unit Circle and Trigonometric Ratios

- The unit circle helps visualize the relationship between sine and cosine for any angle.
- Sine corresponds to the y-coordinate, cosine to the x-coordinate.

Pythagorean Identity

- Fundamental in deriving unknown trigonometric ratios when one ratio is known.
- Ensures the relationship between sine and cosine remains consistent.

Quadrant Considerations

- The sign of trigonometric functions depends on the quadrant.
- Since the problem specifies Quadrant I, both sine and cosine are positive, simplifying the calculation.

Rationalizing Denominators

- A common step to express the answer in a simplified, rationalized form.
- Ensures the radical in the denominator is eliminated for clarity and standardization.

Features and Benefits of the Method

Features:

- Uses fundamental identities and the unit circle concept.

- Emphasizes the importance of quadrant-based sign determination.
- Demonstrates algebraic manipulation to simplify and rationalize expressions.

Pros:

- Clear logical structure makes the solution accessible.
- Rationalization ensures the answer conforms to standard mathematical presentation.
- Reinforces understanding of relationships among trigonometric functions.

Cons:

- Assumes familiarity with identities and the unit circle.
- Might require additional practice for those not comfortable with algebraic manipulation.

Additional Examples and Practice

Example 1: Find \(\tan \theta\) given \(\sin \theta = \frac{\sqrt{2}}{2}\) and \(\theta\) in Quadrant I.

Solution:

- \(\sin \theta = \frac{\sqrt{2}}{2}\)
- Use Pythagorean identity:

\[\cos^2 \theta = 1 - \sin^2 \theta = 1 - \frac{1}{2} = \frac{1}{2} \]

- \(\cos \theta = \frac{\sqrt{2}}{2}\)

- \(\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1 \)

Answer:

\[\boxed{\tan \theta = 1} \]

Example 2: Given \(\sin \theta = \frac{3}{5}\) and \(\theta\) in Quadrant I, find \(\tan \theta\).

Solution:

- Find \(\cos \theta\):

$$\cos^2 \theta = 1 - \left(\frac{3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{16}{25}$$

- $\cos \theta = \frac{4}{5}$

- $\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{3/5}{4/5} = \frac{3}{4}$

Answer:

$$\boxed{\tan \theta = \frac{3}{4}}$$

Common Pitfalls and Tips

- Ignoring the Quadrant: Always check the quadrant to determine the signs of sine and cosine.
- Forgetting to Rationalize: When dealing with radicals in denominators, always rationalize to avoid fractions with radicals.
- Misapplying the Pythagorean Identity: Ensure correct substitution to avoid errors in calculating the missing ratio.
- Assuming Values in Other Quadrants: Remember that sine and cosine signs differ across quadrants, affecting the final answer.

Summary and Conclusion

Finding the exact value of tangent when given sine involves a systematic approach rooted in the fundamental identities of trigonometry. By leveraging the Pythagorean identity, understanding the unit circle, and considering the quadrant, one can accurately determine $\tan \theta$. The process highlights the interconnectedness of trigonometric functions and underscores the importance of algebraic manipulation, including rationalization, for presenting solutions in a clear and standard form. This method is not only essential for solving textbook problems but also serves as a foundation for more advanced applications in calculus, physics, and engineering.

In conclusion, given that $\sin \theta = 1/2$ and θ is in Quadrant I, the exact value of $\tan \theta$ is $\frac{\sqrt{3}}{3}$. Mastery of these techniques enhances problem-solving skills and deepens understanding of the elegant relationships within trigonometry.

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