Find The General Solutions Of The Following Differential Equations Using D-operator Methods: (D^2-5D+6)y=e^-2x

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Introduction to the D-Operator Method

Understanding differential equations is fundamental in mathematical analysis, especially when dealing with linear differential equations with constant coefficients. One powerful approach to solving these equations is the D-operator method, which leverages the algebraic properties of the differential operator D, defined as $D = \frac{d}{dx}$. This method simplifies the process of finding solutions by treating differentiation as algebraic operations, enabling us to factor differential operators similarly to polynomial expressions.

In this article, we will explore how to find the general solution of the differential equation:

$$
(D^2 - 5D + 6) y = e^{-2x}
$$

using the D-operator method. We will first analyze the associated homogeneous equation, then apply the method of annihilators and the inverse operator to find particular solutions.

Understanding the Differential Operator and Its Properties

The Differential Operator D

- The operator D is defined as $D = \frac{d}{dx}$.
- When applied to a function y, $D y = y'$.
- Higher powers of D denote higher derivatives: $D^2 y = y''$, etc.

Factorization of the Differential Operator

- The given differential operator is quadratic:

$$
D^2 - 5D + 6
$$
- This can be factored algebraically:
$$
D^2 - 5D + 6 = (D - 2)(D - 3)
$$
- This factorization is crucial for solving the homogeneous equation and analyzing the operator.

Solving the Homogeneous Differential Equation

Formulating the Homogeneous Equation

- The associated homogeneous differential equation is:
$$
(D^2 - 5D + 6) y = 0
$$
- Using the factorization:
$$
(D - 2)(D - 3) y = 0
$$

Finding the Homogeneous Solution y_h

- The roots of the characteristic equation are $r = 2$ and $r = 3$.
- The general homogeneous solution is:
$$
y_h = C_1 e^{2x} + C_2 e^{3x}
$$
where C_1, C_2 are arbitrary constants.

Applying the D-Operator Method to Find a Particular Solution

Rewriting the Equation Using the D-Operator

- The original inhomogeneous differential equation:
$$
(D^2 - 5D + 6) y = e^{-2x}
$$

\]
- Using the factorization:
\[
(D - 2)(D - 3) y = e^{-2x}
\]

Understanding the Right-Hand Side \(e^{-2x} \)

- The right side is an exponential function, which suggests trying a particular solution involving \(e^{-2x} \).
- Since \(e^{-2x} \) is not a solution to the homogeneous equation (roots are 2 and 3, not -2), we can proceed directly.

Using the Inverse Operator to Find a Particular Solution

- Rewrite as:
\[
y = (D - 2)^{-1} (D - 3)^{-1} e^{-2x}
\]
- This involves applying inverse operators sequentially.

Calculating the Inverse Operators on \(e^{-2x} \)

Inverse Operator \((D - a)^{-1} \)

- The inverse of \((D - a) \) when acting on \(e^{mx} \) is:

\[
(D - a)^{-1} e^{mx} = \frac{1}{m - a} e^{mx}
\]
provided \(m \neq a \).

- Applying this to \(e^{-2x} \):
\[
(D - a)^{-1} e^{-2x} = \frac{1}{-2 - a} e^{-2x}
\]

Sequential Application of Inverse Operators

- First, apply \((D - 3)^{-1} \) on \(e^{-2x} \):
\[

$$
(D - 3)^{-1} e^{-2x} = \frac{1}{-2 - 3} e^{-2x} = \frac{1}{-5} e^{-2x} = -\frac{1}{5} e^{-2x}
$$

- Next, apply $(D - 2)^{-1}$ on the result:

$$
(D - 2)^{-1} \left(-\frac{1}{5} e^{-2x}\right) = -\frac{1}{5} \times \frac{1}{-2 - 2} e^{-2x} = -\frac{1}{5} \times \frac{1}{-4} e^{-2x} = \frac{1}{20} e^{-2x}
$$

Constructing the Particular Solution y_p

- The particular solution is:

$$
y_p = \frac{1}{20} e^{-2x}
$$

Formulating the General Solution

Combining Homogeneous and Particular Solutions

- The general solution to the differential equation is the sum of the homogeneous solution and a particular solution:

$$
y = y_h + y_p = C_1 e^{2x} + C_2 e^{3x} + \frac{1}{20} e^{-2x}
$$

Summary and Final Remarks

- The D-operator method simplifies the process of solving linear differential equations with constant coefficients by algebraic manipulation.
- Factoring the differential operator into linear factors allows straightforward determination of homogeneous solutions.
- Applying inverse operators sequentially to the right-hand side function yields particular solutions, especially when dealing with exponential nonhomogeneous terms.
- The approach underscores the power of operator algebra in differential equations, providing a systematic method that can be extended to more complex cases.

Conclusion

Using the D-operator method, we've efficiently derived the general solution to the differential equation:

$$
(D^2 - 5D + 6) y = e^{-2x}
$$

which is:

$$
\boxed{
y = C_1 e^{2x} + C_2 e^{3x} + \frac{1}{20} e^{-2x}
}
$$

This method not only simplifies the solving process but also enhances understanding of the underlying algebraic structure of differential operators. It is a valuable technique for engineers, mathematicians, and scientists working with linear differential equations and can be adapted to various types of nonhomogeneous terms.

Frequently Asked Questions

What is the differential equation given in the problem?

The differential equation is $(D^2 - 5D + 6)y = e^{-2x}$.

How do you find the general solution of the homogeneous differential equation associated with $(D^2 - 5D + 6)y = e^{-2x}$?

Solve the characteristic equation $r^2 - 5r + 6 = 0$ to find roots $r = 2$ and $r = 3$. The homogeneous solution is $y_h = C_1 e^{2x} + C_2 e^{3x}$.

What is the form of the particular solution for the differential equation using D-operator methods?

Since the nonhomogeneous term is e^{-2x} and e^{-2x} is not a solution to the homogeneous equation, assume a particular solution of $y_p = A e^{-2x}$.

How do you apply the D-operator to find the particular solution in this context?

Rewrite the differential operator as $(D^2 - 5D + 6)$, then apply it to $y_p = A e^{-2x}$ to solve for A by substituting into the equation.

What is the value of the operator $(D^2 - 5D + 6)$ applied to $y_p = A e^{-2x}$?

Applying the operator: $(D^2 - 5D + 6)A e^{-2x} = A[(-2)^2 - 5(-2) + 6] e^{-2x} = A[4 + 10 + 6] e^{-2x} = 20A e^{-2x}$.

How do you determine the coefficient A for the particular solution?

Set the operator applied to y_p equal to the right side: $20A e^{-2x} = e^{-2x}$. Solving gives $A = 1/20$.

What is the complete general solution to the differential equation?

The general solution is $y = C_1 e^{2x} + C_2 e^{3x} + (1/20) e^{-2x}$.

Why is the particular solution assumed to be $y_p = A e^{-2x}$ in this method?

Because e^{-2x} is not a solution to the homogeneous equation, so assuming $y_p = A e^{-2x}$ avoids duplication with the homogeneous solution and simplifies the calculation.

Additional Resources

Differential Equations — Unlocking the Power of D-Operator Methods for General Solutions

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Introduction

When it comes to solving differential equations, the methods employed can significantly influence the efficiency and clarity of the solution process. Among the various techniques available, the D-operator method (also known as the differential operator method) stands out for its elegance and systematic approach, especially when dealing with linear differential equations with constant coefficients. This article delves deeply into the application of D-operator methods to find the general solution of the differential equation:

$$
(D^2 - 5D + 6) y = e^{-2x}
$$

By exploring this example thoroughly, we aim to demonstrate not only how the D-operator method simplifies solving such equations but also provide insights into the underlying concepts, step-by-step procedures, and practical considerations.

Understanding the D-Operator Method

What is the D-Operator?

The D-operator, denoted by D, represents differentiation with respect to x:

$$
D = \frac{d}{dx}
$$

Using this notation, differential equations with constant coefficients can be expressed as algebraic equations in terms of D. For example, a second-order linear differential operator:

$$
D^2 - 5D + 6
$$

acts on a function y as:

$$
(D^2 - 5D + 6) y = 0
$$

which is equivalent to the differential equation:

$$
y'' - 5 y' + 6 y = 0
$$

The key advantage of the D-operator approach is that it allows us to manipulate differential equations algebraically, akin to polynomial equations, simplifying the process of finding solutions.

Why Use D-Operator Methods?

- Systematic Approach: It offers a clear, algebraic pathway to solutions.
- Ease of Handling Nonhomogeneous Terms: By taking reciprocals or factors, it becomes straightforward to manage particular solutions.
- Factorization Capabilities: It facilitates the factorization of differential operators, aiding in finding homogeneous solutions.

Step-by-Step Solution of the Equation

The given differential equation is:

$$
(D^2 - 5D + 6) y = e^{-2x}
$$

Our goal is to find the general solution, which combines the complementary (homogeneous) solution and particular solution.

---

1. Find the Complementary Solution y_c

The first step involves solving the associated homogeneous equation:

$$
(D^2 - 5D + 6) y = 0
$$

which in standard form is:

$$
y'' - 5 y' + 6 y = 0
$$

Characteristic Equation:

Expressed via the D-operator:

$$
D^2 - 5D + 6 = 0
$$

The roots of the characteristic polynomial are found by factoring:

$$
D^2 - 5D + 6 = (D - 2)(D - 3)
$$

So, the roots are:

$$
r_1 = 2, \quad r_2 = 3
$$

Complementary Solution:

$$
y_c = C_1 e^{2x} + C_2 e^{3x}
$$

where C_1 and C_2 are arbitrary constants.

---

2. Find the Particular Solution y_p Using D-Operator Methods

The nonhomogeneous part is e^{-2x}. To find a particular solution, we express the differential operator in terms of factors:

$$
D^2 - 5D + 6 = (D - 2)(D - 3)
$$

\]

The D-operator approach suggests considering the inverse operator acting on the right-hand side:

$$
y_p = \left[(D - 2)(D - 3) \right]^{-1} e^{-2x}
$$

Key insight: To find y_p, we need to "divide" the nonhomogeneous term e^{-2x} by the differential operator.

---

3. Applying the D-Operator Inverse

The inverse of the operator $(D - a)$, when acting on e^{ax}, is straightforward because:

$$
(D - a) e^{ax} = 0
$$

and

$$
(D - a)^{-1} e^{ax} = \frac{e^{ax}}{0}
$$

which is undefined. However, in the context of particular solutions, we interpret the inverse operator as an integral operator or by using the method of undetermined coefficients / variation of parameters.

Alternatively, considering the D-operator approach involves understanding how the inverse acts on functions not in the kernel. Since e^{-2x} is not a solution of $(D - 2)$, the inverse operator can be seen as:

$$
(D - \alpha)^{-1} f(x) = \int e^{\alpha x} \left(\int e^{-\alpha x} f(x) dx \right) dx
$$

but this process can be complex and is often simplified by using the method of undetermined coefficients or Laplace transforms.

---

4. Simplifying the Process: Direct Factor Approach

Given the factorization:

\[

(D - 2)(D - 3) y = e^{-2x}
\]

we can think of solving this stepwise:

- First, define an auxiliary function:

\[
z = (D - 3) y
\]

then,

\[
(D - 2) z = e^{-2x}
\]

- Next, solve for \(z \):

\[
(D - 2) z = e^{-2x}
\]

which gives:

\[
z' - 2 z = e^{-2x}
\]

This is a first-order linear ODE in \(z \).

---

5. Solving for \(z \)

Step 1: Write the integrating factor

\[
\mu(x) = e^{\int -2 dx} = e^{-2x}
\]

Step 2: Multiply through

\[
e^{-2x} z' - 2 e^{-2x} z = e^{-4x}
\]

which simplifies to:

\[
\frac{d}{dx} \left(e^{-2x} z \right) = e^{-4x}

\]

Step 3: Integrate both sides

$$
e^{-2x} z = \int e^{-4x} dx + C
$$

$$
= -\frac{1}{4} e^{-4x} + C
$$

Thus,

$$
z = e^{2x} \left(-\frac{1}{4} e^{-4x} + C \right) = -\frac{1}{4} e^{-2x} + C e^{2x}
$$

Now, recall that:

$$
z = (D - 3) y = y' - 3 y
$$

---

6. Solving for y

The equation:

$$
y' - 3 y = z = -\frac{1}{4} e^{-2x} + C e^{2x}
$$

is a first-order linear ODE in y. To find y_p, focus on the particular part (ignoring the term involving C), which pertains to the homogeneous solution).

The integrating factor:

$$
\mu(x) = e^{-3x}
$$

Multiply through:

$$
e^{-3x} y' - 3 e^{-3x} y = e^{-3x} z
$$

$$
\frac{d}{dx} \left(e^{-3x} y \right) = e^{-3x} z

\]

Substitute \(z \):

\[
\frac{d}{dx} \left(e^{-3x} y \right) = e^{-3x} \left(-\frac{1}{4} e^{-2x} + C e^{2x} \right)
\]

which simplifies to:

\[
\frac{d}{dx} \left(e^{-3x} y \right) = -\frac{1}{4} e^{-5x} + C e^{-x}
\]

Integrate both sides:

\[
e^{-3x} y = \int \left(-\frac{1}{4} e^{-5x} + C e^{-x} \right) dx + K
\]

where \(K \) is an arbitrary constant. Focus on the particular solution:

\[
e^{-3x} y_p = -\frac{1}{4} \int e^{-5x} dx + C \int e^{-x} dx
\]

Compute the integrals:

\[
\int e^{-5x} dx = -\frac{1}{5} e^{-5x}
\]

\[
\int e^{-x} dx = - e^{-x}
\]

Thus,

\[
e^{-3x} y_p = -\frac{1}{4} \left(- \frac{1}{5} e^{-5x} \right) + C (- e^{-x}) + \text{constant}
\]

\[
= \frac{1}{20

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