In Exercises 7-12, Show That ? Is An Eigenvalue Of A And Find One Eigenvector Corresponding To This Eigenvalue.

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Introduction to Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors are fundamental concepts in linear algebra with numerous applications across science, engineering, and mathematics. They help us understand the intrinsic properties of matrices, especially regarding their behavior under linear transformations. When analyzing a matrix A, one of the common tasks is to determine its eigenvalues – scalar values λ such that there exists a non-zero vector v satisfying $A v = \lambda v$. The vector v associated with each eigenvalue is called the eigenvector.

Understanding how to verify that a certain scalar λ is an eigenvalue of a matrix A and finding a corresponding eigenvector are crucial skills in linear algebra. These tasks often involve solving characteristic equations derived from the matrix A.

Step-by-Step Approach to Show That λ Is an Eigenvalue

1. Recall the Definition of Eigenvalues and Eigenvectors

An eigenvalue λ of a matrix A is a scalar for which there exists a non-zero vector v satisfying:

$$
A v = \lambda v
$$

\]

This can be rewritten as:

$$
(A - \lambda I) v = 0
$$

where I is the identity matrix of the same size as A.

2. Formulate the Eigenvalue Equation

To verify whether a given scalar λ is an eigenvalue:

- Construct the matrix $A - \lambda I$.
- Determine whether the homogeneous system $(A - \lambda I) v = 0$ has a non-trivial solution.

Since the trivial solution $v = 0$ always exists, to have a non-trivial solution, the matrix $A - \lambda I$ must be singular, i.e., its determinant must be zero:

$$
\det(A - \lambda I) = 0
$$

This equation is called the characteristic equation, and solving it yields all eigenvalues of A.

3. Compute the Characteristic Polynomial and Solve for λ

- Calculate $\det(A - \lambda I)$.
- Expand this determinant to obtain a polynomial in λ, called the characteristic polynomial.
- Solve the polynomial equation $\det(A - \lambda I) = 0$ for λ. The roots are the eigenvalues.

4. Verify the Given λ

- Once you have the candidate eigenvalue λ, substitute it back into $A - \lambda I$.
- Confirm that the matrix $A - \lambda I$ is singular (determinant zero).
- Confirm that the homogeneous system $(A - \lambda I) v = 0$ has non-trivial solutions, indicating that λ is indeed an eigenvalue.

Finding an Eigenvector Corresponding to \(\lambda\)

1. Set Up the System \((A - \lambda I) v = 0\)

- With \(\lambda\) verified as an eigenvalue, write down the matrix \(A - \lambda I\).
- Formulate the homogeneous system of equations.

2. Solve the System for \(v\)

- Use methods such as Gaussian elimination or matrix row reduction to find the null space of \(A - \lambda I\).
- The solutions to this system form the eigenspace associated with \(\lambda\).

3. Choose a Non-Zero Vector as Eigenvector

- Select any non-zero solution vector \(v\) from the null space.
- This vector is an eigenvector corresponding to \(\lambda\).

4. Normalize if Necessary

- For convenience, sometimes eigenvectors are normalized to have unit length.
- This step is optional but often useful for standardization.

Practical Example

Suppose you are given a matrix:

\[
A = \begin{bmatrix}
4 & 1 \\
2 & 3

\end{bmatrix}
\]

and a scalar candidate \(\lambda = 5\). You are tasked with verifying whether \(\lambda=5\) is an eigenvalue and finding the corresponding eigenvector.

Step 1: Form \(A - 5 I\)

\[
A - 5 I = \begin{bmatrix}
4 - 5 & 1 \\
2 & 3 - 5
\end{bmatrix}
= \begin{bmatrix}
-1 & 1 \\
2 & -2
\end{bmatrix}
\]

Step 2: Check the determinant

\[
\det(A - 5 I) = (-1)(-2) - (1)(2) = 2 - 2 = 0
\]

Since the determinant is zero, \(\lambda=5\) is indeed an eigenvalue.

Step 3: Find the eigenvector

Solve \((A - 5 I) v = 0\):

\[
\begin{bmatrix}
-1 & 1 \\
2 & -2
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

From the first row:

\[
-1 v_1 + 1 v_2 = 0 \Rightarrow v_2 = v_1
\]

The second row:

\[
2 v_1 - 2 v_2 = 0 \Rightarrow 2 v_1 = 2 v_2 \Rightarrow v_2 = v_1
\]

Both equations are consistent, and the eigenvector corresponding to \(\lambda=5\) can be any non-zero multiple of:

\[
v = \begin{bmatrix}
1 \\
1
\end{bmatrix}
\]

Summary and Key Takeaways

- To verify if a scalar \(\lambda\) is an eigenvalue of matrix \(A\), compute \(\det(A - \lambda I)\). If it equals zero, \(\lambda\) is an eigenvalue.
- The characteristic polynomial \(\det(A - \lambda I)\) provides all eigenvalues upon solving.
- The eigenvector corresponding to \(\lambda\) is found by solving \((A - \lambda I) v = 0\).
- Eigenvectors are determined up to scalar multiples, and choosing a convenient non-zero solution is standard practice.
- These procedures are fundamental for understanding matrix behavior, diagonalization, and various applications such as stability analysis, quantum mechanics, and principal component analysis.

Additional Tips and Considerations

- Always verify the eigenvalue by substitution before proceeding to find the eigenvector.

- When solving \((A - \lambda I) v = 0\), look for free variables to parametrize the null space.
- Eigenvectors corresponding to different eigenvalues are linearly independent, which is crucial in diagonalization.
- In higher dimensions, computational tools or software (like MATLAB, NumPy, or WolframAlpha) can facilitate calculations.
- For repeated eigenvalues, check for the algebraic and geometric multiplicities to understand the eigenstructure fully.

This comprehensive approach ensures a clear understanding of verifying eigenvalues and calculating corresponding eigenvectors, foundational skills for advanced studies and applications in linear algebra.

Frequently Asked Questions

In Exercises 7-12, how do you verify that a given scalar ? is an eigenvalue of matrix A?

To verify that ? is an eigenvalue of matrix A, you need to check if the equation (A - ?I) v = 0 has a non-trivial solution, which involves computing the determinant det(A - ?I) and confirming it equals zero.

What is the process to find an eigenvector corresponding to a specific eigenvalue ? of matrix A?

Once you confirm that ? is an eigenvalue, substitute it into (A - ?I) and solve the homogeneous system (A - ?I) v = 0 for v, which yields the eigenvectors associated with ?.

Why is solving the system (A - ?I) v = 0 essential in Exercises 7-12 when showing ? is an eigenvalue?

Because the existence of a non-trivial solution v to (A - ?I) v = 0 confirms that ? is an eigenvalue, with v as its corresponding eigenvector.

Can an eigenvalue ? have multiple linearly

independent eigenvectors? How does this relate to exercises 7-12?

Yes, an eigenvalue ? can have multiple linearly independent eigenvectors, forming an eigenspace. In exercises 7-12, after finding one eigenvector, you can look for additional vectors to understand the eigenspace's dimension.

What are common methods to find eigenvectors once an eigenvalue ? is identified in exercises 7-12?

Common methods include solving (A - ?I) v = 0 using row reduction or Gaussian elimination, and expressing the solution in parametric form to identify eigenvectors corresponding to ?.

Additional Resources

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Introduction: Understanding Eigenvalues and Eigenvectors

Mathematics often presents us with abstract concepts that underpin many practical applications—from engineering and physics to computer science and data analysis. Among these foundational ideas are eigenvalues and eigenvectors, which play a crucial role in understanding linear transformations, stability analysis, and dimensionality reduction techniques like Principal Component Analysis (PCA).

In simple terms, consider a square matrix \(A \) representing a linear transformation. An eigenvalue \(\lambda \) of \(A \) is a scalar such that there exists a non-zero vector \(\mathbf{v} \) (the eigenvector) satisfying:

\[
A \mathbf{v} = \lambda \mathbf{v}
\]

This equation indicates that when the linear transformation \(A \) acts on \(\mathbf{v} \), the result is simply \(\mathbf{v} \) scaled by \(\lambda \). Recognizing eigenvalues and eigenvectors of a matrix is essential for understanding its behavior, especially in systems where stability and modes of operation are critical.

The central task in the exercises (7-12) is to verify whether a specific scalar (denoted as "?") is an eigenvalue of a given matrix \(A \), and if so, to find an eigenvector associated with it. This process combines algebraic techniques with insightful interpretation, offering a window into the matrix’s intrinsic properties.

Theoretical Foundations: How to Show That a Scalar Is an Eigenvalue

The Eigenvalue Equation

To determine if a scalar λ (in this case, "?") is an eigenvalue of matrix A, we rely on the fundamental eigenvalue equation:

$$
A \mathbf{v} = \lambda \mathbf{v}
$$

Rearranged, this becomes:

$$
(A - \lambda I) \mathbf{v} = \mathbf{0}
$$

where I is the identity matrix of the same size as A. This equation suggests that $\mathbf{v}$ is a non-zero vector in the null space (kernel) of $(A - \lambda I)$. For a non-trivial solution (non-zero $\mathbf{v}$) to exist, the matrix $(A - \lambda I)$ must be singular, i.e., its determinant must be zero:

$$
\det(A - \lambda I) = 0
$$

This determinant equation is called the characteristic equation, and solving it yields the eigenvalues of A.

Step-by-Step Process

1. Write Down the Matrix A:
Obtain the explicit form of matrix A from the problem statement.

2. Form the Matrix $A - \lambda I$:
Subtract λ times the identity matrix from A.

3. Compute the Determinant $\det(A - \lambda I)$:
Calculate the determinant as a polynomial in λ.

4. Solve the Characteristic Equation:
Find the roots λ of the polynomial. These roots are the eigenvalues.

5. Identify If "?" Is an Eigenvalue:
Check whether "?" satisfies the characteristic polynomial.

Practical Example: Verifying if "?" Is an Eigenvalue

Suppose A is a 3x3 matrix:

$$
A = \begin{bmatrix}
2 & 1 & 0 \\
0 & 3 & 0 \\
0 & 0 & 4
\end{bmatrix}
$$

And the scalar in question is $\lambda = ?$.

Step 1: Form $A - ? I$

$$
A - ? I = \begin{bmatrix}
2 - ? & 1 & 0 \\
0 & 3 - ? & 0 \\
0 & 0 & 4 - ?
\end{bmatrix}
$$

Step 2: Calculate the determinant

$$
\det(A - ? I) = (2 - ?) \times (3 - ?) \times (4 - ?)
$$

Step 3: Set the determinant to zero and solve

$$
(2 - ?) (3 - ?) (4 - ?) = 0
$$

This gives three possible eigenvalues:

$$
? = 2, \quad ? = 3, \quad ? = 4
$$

Now, if "?" in the problem matches any of these values, then "?" is an eigenvalue of A.

Finding the Corresponding Eigenvector

Once an eigenvalue λ is confirmed, the next step is to find an eigenvector $\mathbf{v}$ satisfying:

$$
(A - \lambda I) \mathbf{v} = \mathbf{0}
$$

Procedure:

1. Substitute λ into $A - \lambda I$.
2. Solve the homogeneous system:
Find the null space of $(A - \lambda I)$.

3. Express the general solution:
Choose free variables to parametrize the solution.

4. Identify a non-zero eigenvector:
Select specific values for free variables to obtain a concrete eigenvector.

Worked Example: Eigenvector for $\lambda = 3$

Continuing with our previous example, suppose $\lambda = 3$.

Step 1: Form $A - 3 I$

$$
A - 3 I = \begin{bmatrix}
-1 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

Step 2: Solve $(A - 3 I) \mathbf{v} = \mathbf{0}$

Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$.

The system of equations:

- $-v_1 + v_2 = 0 \Rightarrow v_2 = v_1$
- $0 = 0$ (second row, no information)
- $v_3 = 0$

Step 3: Write the solution

$$
v_2 = v_1, \quad v_3 = 0
$$

Choose $v_1 = 1$, then:

$$
\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}
$$

This is an eigenvector corresponding to $\lambda = 3$.

Connecting to Exercises 7-12

The exercises challenge students to apply this methodology to different matrices and scalar values, verifying if "?" is an eigenvalue and, if so, extracting an eigenvector. The process emphasizes the importance of determinant calculations, solving homogeneous systems, and interpreting the results.

Common Pitfalls and Tips

- Incorrect determinant computation: Always double-check calculations, especially for larger matrices.
- Misidentifying eigenvalues: Remember that eigenvalues are roots of the characteristic polynomial; not every scalar "?" will satisfy the characteristic equation.
- Choosing eigenvectors: Any non-zero solution to $(A - \lambda I)\mathbf{v} = 0$ is valid; often, selecting a simple scalar multiple clarifies the eigenvector.

Broader Applications: Why Eigenvalues Matter

Eigenvalues and eigenvectors are more than academic exercises—they serve as the backbone of numerous real-world applications:

- Mechanical Systems & Stability:
Eigenvalues determine the stability of equilibrium points in dynamic systems. For example, in control systems, eigenvalues with negative real parts indicate stability.

- Quantum Mechanics:
The energy levels of a quantum system are eigenvalues of the Hamiltonian operator.

- Data Science & Machine Learning:
Techniques like PCA rely on eigenvectors and eigenvalues to identify principal components, reducing high-dimensional data to its most informative features.

- Vibrations & Modal Analysis:

Engineers analyze vibrational modes of structures by computing eigenvalues and eigenvectors of stiffness and mass matrices.

- Graph Theory & Network Analysis:
Eigenvalues of adjacency matrices reveal properties like connectivity and community structures.

Concluding Remarks

The exercises from 7-12 serve as a practical guide to mastering the fundamental process of identifying eigenvalues and eigenvectors. While the calculations can sometimes be algebraically intensive, the underlying principles are straightforward: find the roots of the characteristic polynomial and solve the associated homogeneous system.

Understanding these concepts enhances one's ability to analyze complex systems, predict their behavior, and harness their properties for innovative solutions across diverse fields. Whether tackling a simple matrix problem or modeling phenomena in physics or data science, the skills developed through these exercises lay a robust foundation for further exploration in linear algebra and beyond.

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