Please Answer All Parts Solve The Homogeneous System. Dx/dt = 4x+5y Dy/dt = -4x-4y C) Solve For One Eigenvector

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When working with systems of differential equations, especially homogeneous linear systems, understanding how to find eigenvalues and eigenvectors is crucial. These mathematical tools allow us to analyze the behavior of solutions over time, such as stability, oscillation, or divergence. In this article, we will explore the process of solving the homogeneous system:

$$
\begin{cases}
\frac{dx}{dt} = 4x + 5y \\
\frac{dy}{dt} = -4x - 4y
\end{cases}
$$

and specifically focus on solving for one eigenvector associated with the system's eigenvalues. This comprehensive guide will walk you through the step-by-step method, including calculating eigenvalues, finding eigenvectors, and interpreting the results.

Understanding the Homogeneous System of Differential Equations

A homogeneous system of differential equations can be expressed in matrix form as:

$$
\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}
$$

where A is a 2x2 matrix representing the coefficients of the system. For our specific system:

\[
A = \begin{bmatrix}

4 & 5 \\
-4 & -4
\end{bmatrix}
\]

The goal is to find solutions of the form:

\[
\mathbf{x}(t) = \mathbf{v} e^{\lambda t}
\]

where \(\lambda\) is an eigenvalue of \(A\), and \(\mathbf{v}\) is the corresponding eigenvector. To find these, we need to:

1. Determine the eigenvalues \(\lambda\) by solving the characteristic equation.
2. For each eigenvalue, solve \((A - \lambda I)\mathbf{v} = 0 \) to find eigenvectors.

Step 1: Find the Eigenvalues

The eigenvalues are roots of the characteristic polynomial:

\[
\det(A - \lambda I) = 0
\]

where \(I\) is the identity matrix. Compute:

\[
A - \lambda I = \begin{bmatrix}
4 - \lambda & 5 \\
-4 & -4 - \lambda
\end{bmatrix}
\]

The determinant is:

\[
(4 - \lambda)(-4 - \lambda) - (5)(-4) = 0
\]

Let's expand and simplify this:

\[
(4 - \lambda)(-4 - \lambda) + 20 = 0
\]

Expanding:

$$
(4)(-4 - \lambda) - \lambda(-4 - \lambda) + 20 = 0
$$

which simplifies to:

$$
-16 - 4\lambda + 4\lambda + \lambda^2 + 20 = 0
$$

Note that $-4\lambda + 4\lambda = 0$, so the equation reduces to:

$$
-16 + \lambda^2 + 20 = 0
$$

$$
\lambda^2 + 4 = 0
$$

Thus, the eigenvalues are:

$$
\lambda^2 = -4 \implies \lambda = \pm 2i
$$

Eigenvalues:

$$
\boxed{
\lambda_1 = 2i, \quad \lambda_2 = -2i
}
$$

These are purely imaginary eigenvalues, indicating the system exhibits oscillatory behavior.

Step 2: Find an Eigenvector for One Eigenvalue

Let's focus on $\lambda = 2i$. To find the eigenvector $\mathbf{v}$, solve:

$$
(A - \lambda I)\mathbf{v} = 0
$$

which becomes:

\[
\left(\begin{bmatrix}
4 & 5 \\
-4 & -4
\end{bmatrix} - 2i \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}\right) \mathbf{v} = 0
\]

Simplify:

\[
\begin{bmatrix}
4 - 2i & 5 \\
-4 & -4 - 2i
\end{bmatrix} \mathbf{v} = 0
\]

Let \(\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\). Then, the system is:

\[
\begin{cases}
(4 - 2i) v_1 + 5 v_2 = 0 \\
-4 v_1 + (-4 - 2i) v_2 = 0
\end{cases}
\]

We can pick one equation to solve for the eigenvector components.

Calculating the Eigenvector for \(\lambda = 2i\)

Choose the first equation:

\[
(4 - 2i) v_1 + 5 v_2 = 0
\]

Solve for \(v_2\):

\[
v_2 = - \frac{(4 - 2i)}{5} v_1
\]

To express the eigenvector, choose a convenient \(v_1\), say \(v_1 = 1\):

\[

v_2 = - \frac{4 - 2i}{5}
\]

Thus, an eigenvector corresponding to \(\lambda = 2i\) is:

\[
\mathbf{v} = \begin{bmatrix}
1 \\
- \frac{4 - 2i}{5}
\end{bmatrix}
\]

which can be written explicitly as:

\[
\boxed{
\mathbf{v} = \begin{bmatrix}
1 \\
- \frac{4}{5} + \frac{2i}{5}
\end{bmatrix}
}
\]

Interpreting the Eigenvector and Eigenvalues

The eigenvalues being imaginary suggest the system has oscillatory solutions. The eigenvector provides the initial direction for these oscillations in the phase space. Since the eigenvector contains complex values, the general solution involves real-valued functions obtained through linear combinations of these complex solutions.

The general solution to the system, considering the eigenvalues \(\pm 2i\), can be expressed as:

\[
\mathbf{x}(t) = c_1 \text{Re}(\mathbf{v} e^{2i t}) + c_2 \text{Im}(\mathbf{v} e^{2i t})
\]

which simplifies to sinusoidal functions with specific amplitudes and phases.

Summary of the Process

1. Identify the matrix \(A\):

\[
A = \begin{bmatrix}
4 & 5 \\
-4 & -4
\end{bmatrix}
\]

2. Calculate the eigenvalues:

- Find \(\det(A - \lambda I) = 0\).
- The characteristic polynomial: \(\lambda^2 + 4 = 0\).
- Solutions: \(\lambda = \pm 2i\).

3. Find eigenvectors for each eigenvalue:

- For \(\lambda = 2i\):

\[
v_2 = - \frac{4 - 2i}{5}
\]

- Eigenvector:

\[
\mathbf{v} = \begin{bmatrix}
1 \\
- \frac{4}{5} + \frac{2i}{5}
\end{bmatrix}
\]

4. Interpretation of results:

- The system exhibits oscillatory behavior due to purely imaginary eigenvalues.
- Eigenvectors guide the initial conditions and solution structure.

Conclusion

Solving homogeneous systems of differential equations through eigenvalues and eigenvectors provides deep insight into the system's dynamics. By precisely calculating eigenvalues, especially complex conjugate pairs, and finding corresponding eigenvectors, we can characterize oscillations, stability, and long-term behavior of solutions.

In this detailed exploration, we demonstrated the step-by-step process to solve for an eigenvector associated with a complex eigenvalue. Such techniques are fundamental in advanced differential equations, control systems, and dynamical systems analysis.

Additional Tips for Solving Eigenvectors in Differential Systems

- Always verify the eigenvector by substituting back into $(A - \lambda I)\mathbf{v} = 0$.
- For complex eigenvalues, consider forming the real-valued solutions by combining the real and imaginary parts.
- Use software tools like MATLAB, WolframAlpha, or Python's NumPy for complex eigenvalue calculations to avoid algebraic errors.
- Remember that eigenvectors are determined up to a scalar multiple; choose convenient normalizations if needed.

Meta Description:
Learn how to solve the homogeneous system of differential equations $dx/dt = 4x + 5y$ and $dy/dt = -4x - 4y$. Discover step-by

Frequently Asked Questions

What is the general approach to solving the homogeneous system dx/dt = 4x + 5y and dy/dt = -4x - 4y?

The general approach involves representing the system in matrix form, finding the eigenvalues and eigenvectors of the coefficient matrix, and then using these to construct the general solution.

How do you set up the matrix for the homogeneous system dx/dt = 4x + 5y and dy/dt = -4x - 4y?

The coefficient matrix is set up as $A = \begin{bmatrix} 4 & 5 \\ -4 & -4 \end{bmatrix}$ (written [[4, 5], [-4, -4]]), where each row corresponds to the coefficients of x and y in the equations.

How do you find the eigenvalues of the matrix A = [[4, 5], [-4, -4]]?

Eigenvalues are found by solving the characteristic equation $\det(A - \lambda I) = 0$, which leads to $\lambda^2 = 0$, giving $\lambda = 0$ as a repeated eigenvalue.

What is the significance of the eigenvalue λ = 0 in solving this system?

An eigenvalue of zero indicates the system has a steady-state solution and may have solutions involving polynomial terms or generalized eigenvectors.

How do you find an eigenvector corresponding to $\lambda = 0$ for matrix A?

Substitute $\lambda = 0$ into $(A - \lambda I)$ and solve the resulting homogeneous system to find an eigenvector, e.g., solve $A v = 0$.

Can you demonstrate how to solve for one eigenvector corresponding to $\lambda = 0$?

Yes. For $\lambda = 0$, solve the system: $4x + 5y = 0$ and $-4x - 4y = 0$. From the first, $x = -(5/4) y$. Choose $y = 4$, then $x = -5$. So, an eigenvector is $v = [-5, 4]$.

After finding an eigenvector, how is it used in constructing the solution to the system?

The eigenvector is used to write the general solution as a linear combination involving exponential functions, e.g., $x(t)$ and $y(t)$ scaled by $e^{\lambda t}$ times the eigenvector components.

What are the key steps to fully solve the homogeneous system after finding an eigenvector?

Key steps include finding all eigenvalues and eigenvectors, forming the general solution from these, and applying initial conditions if provided to determine specific solutions.

Additional Resources

Solve The Homogeneous System: A Deep Dive into Eigenvalues and Eigenvectors

In the realm of differential equations, particularly systems describing real-world phenomena such as physics, engineering, and biology, the analysis of homogeneous systems plays a pivotal role. These systems, characterized by their linearity and absence of forcing functions, often lead us to explore eigenvalues and eigenvectors—mathematical tools that simplify the process of finding solutions. This article aims to provide a comprehensive, detailed, and analytical exploration of solving a specific homogeneous linear system:

$$
\begin{cases}
\frac{dx}{dt} = 4x + 5y \\
\frac{dy}{dt} = -4x - 4y
\end{cases}
$$

with a particular focus on solving for one eigenvector associated with the system's eigenvalues.

Understanding the Homogeneous System of Differential Equations

Definition and Significance

A homogeneous system of linear differential equations consists of equations where the derivatives of variables are linear combinations of those variables without any constant or external forcing terms. Mathematically, such systems can be expressed as:

\[
\frac{d\mathbf{x}}{dt} = A \mathbf{x}
\]

where \(\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \) and \(A \) is a constant coefficient matrix.

In our case:

\[
A = \begin{bmatrix}
4 & 5 \\
-4 & -4
\end{bmatrix}
\]

Understanding these systems is crucial because they often serve as the foundation for analyzing stability, oscillations, and long-term behavior of dynamical systems.

Why Eigenvalues and Eigenvectors Matter

The crux of solving such systems lies in converting the coupled differential equations into a form that can be decoupled, typically through eigen-decomposition of matrix \(A \). Eigenvalues provide scalar indicators of the system's growth or decay rates, while eigenvectors define directions in the phase space along which the system's behavior simplifies into exponential solutions.

Step-by-Step Solution Strategy

1. Formulate the System in Matrix Form

The original system can be succinctly written as:

$$
\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
$$

This compact form facilitates the application of linear algebra techniques.

2. Determine the Eigenvalues

Eigenvalues λ are solutions to the characteristic equation:

$$
\det(A - \lambda I) = 0
$$

where I is the identity matrix.

Calculating:

$$
\det \begin{bmatrix}
4 - \lambda & 5 \\
-4 & -4 - \lambda
\end{bmatrix} = 0
$$

which expands to:

$$
(4 - \lambda)(-4 - \lambda) - (5)(-4) = 0
$$

Simplify step-by-step:

$$
(4 - \lambda)(-4 - \lambda) + 20 = 0
$$

Multiplying out:

$$
(4)(-4) + 4(-\lambda) - \lambda(-4) - \lambda^2 + 20 = 0
$$

$$
-16 - 4\lambda + 4\lambda - \lambda^2 + 20 = 0
$$

Notice $-4\lambda + 4\lambda = 0$, simplifying to:

$$
-16 - \lambda^2 + 20 = 0
$$

Combine like terms:

$$
(20 - 16) - \lambda^2 = 0
$$

$$
4 - \lambda^2 = 0
$$

Solve for λ:

$$
\lambda^2 = 4
$$

$$
\lambda = \pm 2
$$

Thus, the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -2$.

3. Find the Eigenvectors

Once eigenvalues are identified, the next step is to find their corresponding eigenvectors.

Calculating Eigenvectors: The Focus on One Eigenvalue

Eigenvector Corresponding to $\lambda = 2$

To find an eigenvector $\mathbf{v}$ associated with $\lambda = 2$, solve:

$$
(A - 2I) \mathbf{v} = 0
$$

Compute $A - 2I$:

$$
A - 2I = \begin{bmatrix}
4 - 2 & 5 \\
-4 & -4 - 2
\end{bmatrix} = \begin{bmatrix}
2 & 5 \\
-4 & -6
\end{bmatrix}
$$

Set up the homogeneous system:

$$
\begin{bmatrix}
2 & 5 \\
-4 & -6
\end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0
\end{bmatrix}
$$

From the first row:

$$
2v_1 + 5v_2 = 0
$$

Express v_1 in terms of v_2:

$$
v_1 = -\frac{5}{2} v_2
$$

Choose $v_2 = 1$ for simplicity (eigenvectors are determined up to scalar multiples):

$$
v_1 = -\frac{5}{2}
$$

Thus, an eigenvector corresponding to $\lambda = 2$ is:

$$
\mathbf{v}_1 = \begin{bmatrix} -\frac{5}{2} \\ 1 \end{bmatrix}
$$

or, scaled for simplicity:

$$
\boxed{
\mathbf{v}_1 = \begin{bmatrix} -5 \\ 2 \end{bmatrix}
}
$$

Interpretation and Significance of the Eigenvector

Eigenvectors reveal the fundamental directions along which the system evolves exponentially, scaled by their respective eigenvalues. For $\lambda = 2$, the positive eigenvalue indicates an exponential growth along the eigenvector $\mathbf{v}_1$. This means that if initial conditions align with this eigenvector, the solution will grow exponentially at a rate dictated by e^{2t}.

The eigenvector itself characterizes the trajectory in the phase space where the system's behavior simplifies into a pure exponential function, making it a cornerstone in analyzing stability, oscillations, and long-term dynamics.

Summary of the Analytical Process

To fully appreciate the solution, it's instructive to summarize the key steps:

- Formulate the matrix representing the system.
- Calculate eigenvalues by solving the characteristic polynomial.
- Determine eigenvectors for each eigenvalue, focusing here on the one for $\lambda = 2$.
- Interpret eigenvalues and eigenvectors in the context of the system's dynamics.

This systematic approach transforms a coupled system of differential equations into manageable parts, enabling explicit solutions and qualitative understanding.

Broader Implications and Applications

Understanding how to solve for eigenvectors in a homogeneous system extends beyond theoretical mathematics—it has practical implications across scientific disciplines:

- Physics: Analyzing oscillatory systems and stability of equilibrium points.
- Engineering: Designing control systems and understanding signal propagation.
- Economics: Modeling dynamic systems like market equilibria.
- Biology: Investigating population dynamics and stability.

By mastering the process of eigenvalue and eigenvector computation, practitioners can predict system behavior, optimize designs, and interpret complex phenomena effectively.

Conclusion

In the landscape of differential equations, solving homogeneous systems via eigenvalues and eigenvectors remains a fundamental technique. Through meticulous computation of eigenvalues and the derivation of eigenvectors—exemplified here for \(\lambda = 2 \)—analysts can decode the intrinsic behavior of complex systems. The eigenvector associated with this eigenvalue not only simplifies the solution process but also provides deep insight into the system's directional tendencies and stability characteristics. This analytical approach underscores the elegance and power of linear algebra in unraveling the dynamics of interconnected variables, reinforcing its central role in applied mathematics and scientific inquiry.

Keywords: homogeneous system, differential equations, eigenvalues, eigenvectors, matrix diagonalization, stability, dynamical systems

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