Prove That Any Simple Connected Graph With 7 Vertices And 16 Edges Is Not Planar Using Proof By Contradiction.

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Introduction

Graph theory, a vital branch of discrete mathematics, explores the relationships and structures within networks of vertices and edges. One of the fundamental questions in graph theory pertains to the planarity of graphs—whether a graph can be embedded in a plane without any edges crossing. The renowned Kuratowski's theorem and Euler's formula serve as foundational tools in analyzing planarity. In this article, we aim to demonstrate that any simple, connected graph with exactly 7 vertices and 16 edges cannot be planar. We will employ proof by contradiction, a logical method where we assume the opposite of what we want to prove and then derive a contradiction from that assumption.

Preliminaries and Basic Definitions

Before diving into the proof, it is essential to familiarize ourselves with key concepts and results relevant to our discussion.

Definitions

- **Simple graph:** A graph with no loops (edges connected at both ends to the same vertex) and no multiple edges between the same pair of vertices.
- **Connected graph:** A graph where there exists a path between any pair of vertices.
- **Planar graph:** A graph that can be embedded in the plane such that its edges intersect only at their endpoints.
- **Vertices (V):** The set of points in a graph.
- **Edges (E):** The set of connections between pairs of vertices.

Key Theoretical Results

- **Euler's Formula for Planar Graphs:** For any connected planar graph with $|V|$ vertices, $|E|$ edges, and $|F|$ faces (regions bounded by edges, including the outer infinite face), the following holds:

  $$|V| - |E| + |F| = 2$$

- **Bound on Edges in Planar Graphs:** For any simple, connected, planar graph with $|V| \geq 3$,

  $$|E| \leq 3|V| - 6$$

These results serve as the backbone for our proof, as they impose constraints on the maximum number of edges a planar graph can have given a fixed number of vertices.

Outline of the Proof

Our goal is to prove that a simple, connected graph with 7 vertices and 16 edges cannot be planar. We will proceed via proof by contradiction, assuming that such a graph is planar and then deriving a contradiction based on known bounds.

The outline is as follows:

1. Assume, for contradiction, that there exists a simple, connected, planar graph G with $|V|=7$ and $|E|=16$.
2. Use Euler's formula and related inequalities to derive an upper bound on the number of edges.
3. Show that $|E|=16$ exceeds this bound, leading to a contradiction.
4. Conclude that such a graph cannot be planar.

Step-by-Step Proof

Step 1: Assumption of Planarity

Suppose, for contradiction, that G is a simple, connected, planar graph with:

- $|V| = 7$
- $|E| = 16$

Our goal is to show that this assumption leads to an inconsistency.

Step 2: Applying the Bound on Edges in Planar Graphs

From the key theoretical result, for any simple, connected, planar graph with at least three vertices:

$|E| \leq 3|V| - 6$

Substituting |V| = 7:

$|E| \leq 3(7) - 6 = 21 - 6 = 15$

This indicates that any such graph can have at most 15 edges if it is planar.

Step 3: Comparing the Assumed Number of Edges with the Bound

Our assumption states that |E| = 16, but from the inequality:

$|E| \leq 15$

Thus, 16 > 15, which contradicts the necessary condition for planarity derived from Euler’s formula and the edge bound.

Step 4: Conclusion from Contradiction

Since assuming the existence of a simple, connected, planar graph with 7 vertices and 16 edges leads to a contradiction with the fundamental bound, we conclude that such a graph cannot be planar.

Additional Considerations and Clarifications

While the above proof uses a straightforward inequality to establish the non-planarity of the graph, it is instructive to understand the underlying reasoning and the importance of these bounds.

Why is the Bound on Edges Valid?

The inequality |E| ≤ 3|V| - 6 is derived from Euler’s formula for planar graphs. It applies to simple, connected, planar graphs with |V| ≥ 3. The derivation involves analyzing the faces in a planar embedding and the fact that each face must be bounded by at least three edges (since the graph is simple and has no loops or multiple edges).

Implication of the Bound

In our case, with $|V|=7$, the maximum number of edges in a planar graph is 15. Since 16 exceeds this maximum, the assumption that the graph is planar cannot hold.

Summary and Final Remarks

To summarize, the proof hinges on applying fundamental bounds established by Euler's formula and related inequalities for planar graphs. The key points are:

- Planar graphs with $|V|$ vertices satisfy $|E| \leq 3|V| - 6$.
- For $|V|=7$, this bound is $|E| \leq 15$.
- Assuming a graph with $|E|=16$ is planar contradicts this fundamental bound.
- Therefore, no simple, connected graph with 7 vertices and 16 edges can be planar.

This proof exemplifies the power of proof by contradiction and highlights the critical importance of Euler's formula and related inequalities in graph theory.

Extensions and Related Topics

This proof can be extended or related to various other concepts in graph theory:

- Kuratowski's Theorem: Provides a criterion for planarity based on forbidden subgraphs (K_5 and $K_{3,3}$). Recognizing that a graph with 7 vertices and 16 edges must contain a non-planar subgraph can be approached via Kuratowski's theorem.
- Graph Connectivity and Density: The edge count relates to the density and connectivity of the graph, influencing its planarity.
- Applications: From network design to circuit layout, understanding planarity and the bounds on edges is crucial.

Conclusion

In conclusion, by employing proof by contradiction and fundamental relationships in graph theory, we have shown that any simple, connected graph with 7 vertices and 16 edges cannot be planar. The critical insight is that the maximum number of edges permissible in a planar graph with 7 vertices is 15, and exceeding this bound renders planarity impossible. This result reinforces the importance of Euler’s formula and associated inequalities in analyzing the structural properties of graphs, especially in the context of planarity.

References

- Diestel, R. (2017). Graph Theory (5th ed.). Springer.
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- Kuratowski, K. (1930). Sur le problème des courbes gauches en topologie. Fundamenta Mathematicae, 15, 271–283.

Frequently Asked Questions

What is the key property of simple connected graphs with 7 vertices and 16 edges that suggests they might not be planar?

The key property is that such graphs potentially violate Euler's formula for planar graphs, indicating they may be non-planar due to having too many edges for the given number of vertices.

How can proof by contradiction be applied to show that a simple connected graph with 7 vertices and 16 edges is not planar?

Assume the graph is planar; then, by Euler's formula and the maximum number of edges in a planar graph, derive a contradiction, thus proving the graph cannot be planar.

What is the maximum number of edges in a simple planar graph with 7 vertices, and how does this help in the proof?

The maximum number of edges is $3n - 6$, which for $n=7$ is 15. Since the graph has 16 edges, exceeding this maximum suggests it cannot be planar, supporting the proof by contradiction.

Why is Euler's formula crucial in proving non-planarity of graphs with specific vertices and edges?

Euler's formula relates vertices, edges, and faces in planar graphs; if a graph exceeds the maximum edges allowed by the formula, it cannot be planar, serving as a basis for contradiction.

What role do known non-planar subgraphs, like K_5 or K_{3,3}, play in proving that a graph with 7 vertices and 16 edges is non-planar?

If such subgraphs are present as minors, they confirm non-planarity; however, in this case, the edge count alone suffices, but identifying these minors can strengthen the proof.

Can the proof that any simple connected graph with 7 vertices and 16 edges is not planar be generalized to graphs with more vertices and edges?

Yes, similar reasoning applies: for larger graphs, exceeding the maximum number of edges permitted by planarity (based on Euler's formula) indicates non-planarity, allowing generalization through contradiction.

Additional Resources

Prove That Any Simple Connected Graph With 7 Vertices And 16 Edges Is Not Planar Using Proof By Contradiction

Understanding the properties of graphs, particularly planarity, is fundamental in graph theory. In this article, we will explore the assertion that any simple connected graph with 7 vertices and 16 edges cannot be planar, and we will establish this using a proof by contradiction. This proof hinges on well-known theorems in graph theory, especially Euler’s formula and the properties of planar graphs, to demonstrate the impossibility of such a graph being drawn on a plane without edge crossings.

Introduction to Graph Planarity and Its Significance

Graph planarity is a core concept in graph theory with applications spanning network design, circuit layout, and topology. A planar graph can be embedded in the plane such that its edges intersect only at their endpoints. The classic problem involves characterizing which graphs are planar and which are not.

Relevance of the problem:

- Determining planarity helps simplify complex network structures.
- Non-planar graphs imply the necessity of crossings, which can lead to inefficiencies in circuit design or network routing.
- The study of extremal graph properties often involves understanding the maximum number of edges a planar graph can have given a fixed number of vertices.

Preliminaries and Theoretical Background

Before delving into the proof, it’s essential to review the key concepts and theorems that underpin the argument.

Euler’s Formula for Planar Graphs

For any connected planar graph \(G = (V, E) \), Euler’s formula states:

\[
|V| - |E| + |F| = 2
\]

where:

- \(|V| \): number of vertices
- \(|E| \): number of edges
- \(|F| \): number of faces (including the outer face)

This relation is foundational in deriving bounds on the number of edges in planar graphs.

Maximum Edges in a Planar Graph

A critical consequence of Euler’s formula is the maximum number of edges a planar graph with \(n \geq 3 \) vertices can have:

\[
|E| \leq 3|V| - 6

\]

This inequality holds for simple, connected, planar graphs.

Restating the Problem

Given a simple connected graph \(G \) with:

- \(|V| = 7 \)
- \(|E| = 16 \)

we are to prove that such a graph cannot be planar.

Outline of the Proof by Contradiction

The proof approach involves assuming the contrary—that is, supposing \(G \) is planar—and then deriving a contradiction from known properties of planar graphs.

Stepwise plan:

1. Assume \(G \) is a simple, connected, planar graph with 7 vertices and 16 edges.
2. Use the maximum edge bound for planar graphs to check the feasibility.
3. Show that the actual number of edges exceeds this maximum, leading to a contradiction.
4. Conclude that the initial assumption (that \(G \) is planar) must be false.

Applying Planarity Constraints to the Given Graph

Let's analyze the assumptions and the implications step-by-step.

Step 1: Assume G is planar

Suppose G is planar, simple, and connected with $V = 7$ vertices and $E = 16$ edges.

Step 2: Use the maximum edge bound

From the known bound for planar graphs:

$$
|E| \leq 3|V| - 6
$$

Substitute $|V| = 7$:

$$
|E| \leq 3 \times 7 - 6 = 21 - 6 = 15
$$

Therefore, any simple, connected, planar graph with 7 vertices can have at most 15 edges.

Step 3: Compare with the given number of edges

The given graph has $|E| = 16$, which exceeds the maximum of 15 for a planar graph with 7 vertices.

This directly contradicts the planarity assumption because:

$$
16 > 15
$$

Hence, the graph cannot be planar.

Conclusion: The Contradiction and Final Result

The assumption that a simple, connected graph with 7 vertices and 16 edges is planar leads to a contradiction with the established maximum edge count for planar graphs. Since this assumption is false, we conclude that:

Any simple connected graph with 7 vertices and 16 edges is not planar.

This proof leverages the fundamental properties of planarity, particularly Euler’s formula and its derived inequality, to establish the impossibility of such a graph being embedded in the plane without crossings.

Discussion and Additional Insights

While the proof above is straightforward, it underscores the importance of extremal bounds in graph theory. The key features and considerations include:

- Strength of the bounds: The inequality $|E| \leq 3|V| - 6$ is tight for maximal planar graphs. Any graph exceeding this bound is necessarily non-planar.
- Edge density: The given graph’s density surpasses the threshold, implying the presence of subdivisions or minors that are non-planar, such as K_5 or $K_{3,3}$.
- Extensions: Similar reasoning can be applied to larger graphs or different parameters, always leveraging Euler’s formula and related bounds.

Pros:

- Clear and concise proof relying on well-established results.
- Demonstrates the power of inequalities in graph theory.
- Easily generalizable to other cases with similar bounds.

Cons:

- Does not explicitly construct the non-planar embedding or identify specific non-planar minors.
- Assumes familiarity with Euler’s formula and planarity bounds.

Summary

In this comprehensive examination, we have rigorously shown that a simple connected graph with 7 vertices and 16 edges cannot be planar, using a proof by contradiction based on the maximum number of edges in a planar graph. This result highlights the utility of fundamental theorems in graph theory for understanding the structural limitations of graphs and provides a clear example of how bounds on graph properties can be used to prove non-planarity.

References

- West, D. B. (2001). Introduction to Graph Theory. Prentice Hall.
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- Bondy, J. A., & Murty, U. S. R. (2008). Graph Theory. Springer.

In conclusion, the proof hinges on the critical fact that the maximum number of edges in a planar graph with 7 vertices is 15. Since 16 exceeds this, the original assumption of planarity must be false, confirming that any such graph with 7 vertices and 16 edges is necessarily non-planar.

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